# Cut-elimination and Proof Search for Bi-Intuitionistic Tense Logic

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#### Abstract

We consider an extension of bi-intuitionistic logic with the traditional modalities  $\Diamond$ ,  $\Box$ ,  $\blacklozenge$  and  $\blacksquare$  from tense logic Kt. Proof theoretically, this extension is obtained simply by extending an existing sequent calculus for bi-intuitionistic logic with typical inference rules for the modalities used in display logics. As it turns out, the resulting calculus,  $\mathbf{LBiKt}$ , seems to be more basic than most intuitionistic tense or modal logics considered in the literature, in particular, those studied by Ewald and Simpson, as it does not assume any a priori relationship between the modal operators  $\Diamond$  and  $\Box$ . We recover Ewald's intuitionistic tense logic and Simpson's intuitionistic modal logic by modularly extending  $\mathbf{LBiKt}$  with additional structural rules. The calculus  $\mathbf{LBiKt}$  is formulated in a variant of display calculus, using a form of sequents called nested sequents. Cut elimination is proved for  $\mathbf{LBiKt}$ , using a technique similar to that used in display calculi. As in display calculi, the inference rules of  $\mathbf{LBiKt}$  are "shallow" rules, in the sense that they act on top-level formulae in a nested sequent. The calculus  $\mathbf{LBiKt}$  is ill-suited for backward proof search due to the presence of certain structural rules called "display postulates" and the contraction rules on arbitrary structures. We show that these structural rules can be made redundant in another calculus,  $\mathbf{DBiKt}$ , which uses deep inference, allowing one to apply inference rules at an arbitrary depth in a nested sequent. We prove the equivalence between  $\mathbf{LBiKt}$  and  $\mathbf{DBiKt}$  and outline a proof search strategy for  $\mathbf{DBiKt}$ . We also give a Kripke semantics and prove that  $\mathbf{LBiKt}$  is sound with respect to the semantics, but completeness is still an open problem. We then discuss various extensions of  $\mathbf{LBiKt}$ .

 $\textit{Keywords:} \ \ \text{Intuitionistic logic, modal logic, intuitionistic modal logic, deep inference.}$ 

# 1 Introduction

Intuitionistic logic Int forms a rigorous foundation for many areas of Computer Science via its constructive interpretation and via the Curry-Howard isomorphism between natural deduction proofs and well-typed terms in the  $\lambda$ -calculus. Central to both concerns are syntactic proof calculi with cut-elimination and backwards proof-search for finding derivations automatically.

In traditional intuitionistic logic, the connectives  $\to$  and  $\wedge$  form an adjoint pair in that  $(A \wedge B) \to C$  is valid iff  $A \to (B \to C)$  is valid iff  $B \to (A \to C)$  is valid. Rauszer [21] obtained BiInt by extending Int with a binary connective  $-\!\!\!<$  called "exclusion" which is adjoint to  $\vee$  in that  $A \to (B \vee C)$  is valid iff  $(A -\!\!\!< B) \to C$  is valid iff  $(A -\!\!\!< C) \to B$  is valid. Crolard [4] showed that BiInt has a computational interpretation in terms of continuation passing style semantics. Uustalu and Pinto recently showed that Rauszer's sequent calculus [20] and Crolard's extensions of it

fail cut-elimination, but a nested sequent calculus with cut-elimination [9] and a labelled sequent calculus [17] with cut-free-completeness have been found for BiInt.

The literature on Intuitionistic Modal/Tense Logics (IM/TLs) is vast [6,23] and typically uses Hilbert calculi with algebraic, topological or relational semantics. We omit details since our interest is primarily proof-theoretic. Sequent and natural deduction calculi for IMLs are rarer [13,1,16,3,5,11,7]. Extending them with "converse" modalities like  $\spadesuit$  and  $\blacksquare$  causes cut-elimination to fail as it does for classical modal logic S5 where  $\lozenge$  is a self-converse. Labels [14,23,15] can help but are not purely proof-theoretic since they encode the Kripke semantics.

The closest to our work is that of Sadrzadeh and Dyckhoff [22] who give a cut-free sequent calculus using deep inference for a logic with an adjoint pair of modalities  $(\blacklozenge, \Box)$  plus only  $\land$ ,  $\lor$ ,  $\top$  and  $\bot$ . As all their connectives are "monotonic", cut-elimination presents no difficulties.

Let BiKt be the bi-intuitionistic tense logic obtained by extending BiInt with two pairs of adjoint modalities  $(\lozenge, \blacksquare)$  and  $(\blacklozenge, \square)$ , with no explicit relationship between the modalities of the same colour, namely,  $(\lozenge, \square)$  and  $(\blacklozenge, \blacksquare)$ . The modalities form an adjunction as follows:  $A \to \square B$  iff  $\blacklozenge A \to B$  and  $A \to \blacksquare B$  iff  $\lozenge A \to B$ .

Our shallow inference calculus **LBiKt** is a merger of two sub-calculi for BiInt and Kt derived from Belnap's inherently modular display logic. **LBiKt** has syntactic cut-elimination, but is ill-suited for backward proof search. Our deep inference calculus **DBiKt** is complete with respect to the cut-free fragment of **LBiKt** and is more amenable to proof search as it contains no display postulates and contraction rules.

To complete the picture, we also give a Kripke semantics for BiKt based upon three relations  $\leq$ ,  $R_{\Diamond}$  and  $R_{\Box}$ . The logic BiKt enjoys various desirable properties:

- \* Conservativity: it is a conservative extension of intuitionistic logic Int, dual intuitionistic logic DInt, and bi-intuitionistic logic BiInt;
- \* Classical Collapse: it collapses to classical tense logic by the addition of four structural rules;
- \* **Disjunction Property:** If  $A \vee B$  is a theorem not containing  $\prec$  then A is a theorem or B is a theorem;
- \* **Dual Disjunction Property:** If  $A \wedge B$  is a counter-theorem not containing  $\rightarrow$  then so is A or B;
- \* Independent  $\Diamond$  and  $\Box$ : there is no *a priori* relationship between these connectives

The independence of  $\Diamond$  and  $\square$  is a departure from traditional intuitionistic tense or modal logics, e.g., those considered by Ewald [6] and Simpson [23]. Both Ewald and Simpson allow a form of interdependency between  $\Diamond$  and  $\square$ , expressed as the axiom  $(\Diamond A \to \square B) \to \square (A \to B)$ , which is not derivable in **LBiKt**. However, we shall see in Section 8 that we can recover Ewald's intuitionistic tense logic and Simpson's intuitionistic modal logic by extending **LBiKt** with two structural rules.

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\begin{array}{lll} \tau^-(A) &=& A & \tau^+(A) &=& A \\ \tau^-(X,Y) &=& \tau^-(X) \wedge \tau^-(Y) & \tau^+(X,Y) &=& \tau^+(X) \vee \tau^+(Y) \\ \tau^-(X \triangleright Y) &=& \tau^-(X) & \tau^+(Y) & \tau^+(X \triangleright Y) &=& \tau^-(X) \to \tau^+(Y) \\ \tau^-(\bullet X) &=& \diamond \tau^-(X) & \tau^+(\bullet X) &=& \Box \tau^+(X) \\ \tau(X \triangleright Y) &=& \tau^-(X) \to \tau^+(Y) & \tau^+(\bullet X) &=& \blacksquare \tau^+(X) \end{array}
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Fig. 1. Formula Translation of Nested Sequents

### 2 Nested Sequents

The formulae of BiKt are built from a set Atoms of atomic formulae via the grammar below, with  $p \in Atoms$ :

$$A ::= p \mid \top \mid \bot \mid A \to A \mid A \multimap A \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \mid \blacksquare A \mid \blacklozenge A.$$

A structure is defined by the following grammar, where A is a BiKt formula:

$$X := \emptyset \mid A \mid (X, X) \mid X \triangleright X \mid \circ X \mid \bullet X.$$

The structural connective "," is associative and commutative and  $\emptyset$  is its unit. We always consider structures modulo these equivalences. To reduce parentheses, we assume that "o" and "•" bind tighter than "," which binds tighter than "▷". Thus, we write • $X, Y \triangleright Z$  to mean  $(\bullet(X), Y) \triangleright Z$ .

A nested sequent is a structure of the form  $X \triangleright Y$ . This notion of nested sequents generalises Kashima's nested sequents [12] for classical tense logics, Brünnler's nested sequents [2] and Poggiolesi's tree-hypersequents [18] for classical modal logics. Figure 1 shows the formula-translation of nested sequents. On both sides of the sequent,  $\circ$  is interpreted as a white (modal) operator and  $\bullet$  as a black (tense) operator. Note that however, on the lefthand side of the sequent,  $\triangleright$  is interpreted as exclusion, while on the righthand side, it is interpreted as implication.

The occurrence of a formula A in a structure can have three different polorities: neutral, positive or negative. These are defined inductively below:

- \* The occurrence of A in X is neutral if it does not occur in the scope of the structural connective  $\triangleright$ .
- \* If the occurrence of A in X is neutral then it is positive in  $Y \triangleright X$  and negative in  $X \triangleright Y$  for any structure Y.
- \* If the occurrence of A in X is positive (resp. negative) then it is also positive (resp. negative) in  $X \triangleright Y$ ,  $Y \triangleright X$ , (X,Y), (Y,X),  $\circ X$  and  $\bullet X$ , for any structure Y.

The polarity of a structure occurrence X in another structure Y is defined analogously, substituting the formula occurrence for the structure occurrence X. Note that as a consequence of the overloading of  $\triangleright$  to represent the structural proxies for both  $\rightarrow$  and  $-\!\!<$ , further nesting of a negative context within  $\triangleright$  does not change its polarity.

A context is a structure with a hole or a placeholder []. Contexts are ranged over by  $\Sigma$ []. We write  $\Sigma[X]$  for the structure obtained by filling the hole [] in the context  $\Sigma[]$  with a structure X. The notion of polarities of a context is defined as above, treating the hole in the context as a formula occurrence. We say that a context  $\Sigma[]$  is neutral if the hole [] has neutral polarity, positive if it has positive polarity, and negative if it has negative polarity. Thus, the hole in a neutral context is never under the scope of  $\triangleright$ . We write  $\Sigma^-[]$  to indicate that  $\Sigma[]$  is a negative context and  $\Sigma^+[]$  to indicate that it is a positive context. Intuitively, if one views a nested sequent as a tree (with structural connectives and multisets of formulae as nodes), then a hole in a context is negative if it appears to the left of the closest ancestor node labelled with  $\triangleright$ .

The context  $\Sigma[]$  is *strict* if it has any of the forms:

$$\Sigma'[X \triangleright []]$$
  $\Sigma'[[] \triangleright X]$   $\Sigma'[\circ []]$   $\Sigma'[\bullet []]$ 

Intuitively, in the formation tree of a strict context, the hole must be an immediate child of  $\triangleright$  or  $\circ$  or  $\bullet$ . This notion of strict contexts will be used in later in Section 3.

**Example 2.1** The context  $\bullet([], (X \triangleright Y))$  is a neutral context but  $\bullet(([], X) \triangleright Y)$  is not. Both  $\bullet([], (X \triangleright Y)) \triangleright Z$  and  $\bullet(([], X) \triangleright Y) \triangleright Z$  are negative contexts. The context  $\bullet[] \triangleright Z$  is a strict context but  $\bullet(([], X) \triangleright Y) \triangleright Z$  is not.

# 3 Nested Sequent Calculi

We now present the two nested sequent calculi that we will use in the rest of the paper: a shallow inference calculus **LBiKt** and a deep inference calculus **DBiKt**.

Fig. 2 gives the rules of the shallow inference calculus  $\mathbf{LBiKt}$ . Central to this calculus is the idea that inference rules can only be applied to formulae at the top level of nested sequents, and the structural rules  $s_L$ ,  $s_R$ ,  $\triangleright_L$ ,  $\triangleright_R$ ,  $rp_\circ$  and  $rp_\bullet$ , also called the residuation rules, are used to bring the required sub-structures to the top level. These rules are similar to residuation postulates in display logic, are essential for the cut-elimination proof of  $\mathbf{LBiKt}$ , but contain too much non-determinism for effective proof search. Another issue with proof search in  $\mathbf{LBiKt}$  is the structural contraction rules, which allow contraction on arbitrary structures, not just formulae as in traditional sequent calculi.

**LBiKt** is as a merger of two calculi: the LBiInt calculus [9,19] for the intuitionistic connectives, and the display calculus [8] for the tense connectives.

Note that we use  $\circ$  and  $\bullet$  as structural proxies for the non-residuated pairs  $(\diamondsuit, \Box)$  and  $(\blacklozenge, \blacksquare)$  respectively, whereas Wansing [24] uses only one  $\bullet$  as a structural proxy for the residuated pair  $(\blacklozenge, \Box)$  and recovers  $(\diamondsuit, \blacksquare)$  via classical negation, while Goré [8] uses  $\circ$  and  $\bullet$  as structural proxies for the residuated pairs  $(\diamondsuit, \blacksquare)$  and  $(\blacklozenge, \Box)$  respectively. As we shall see later, our choice allows us to retain the modal fragment  $(\diamondsuit, \Box)$  by simply eliding all rules that contain "black" operators from our deep sequent calculus.

Fig. 3 gives the rules of the deep inference calculus **DBiKt**. Here the inference rules can be applied at any level of the nested sequent, indicated by the use of contexts. Notably, there are no residuation rules; indeed one of the goals of our paper is to show that the residuation rules of **LBiKt** can be simulated by deep inference and propagation rules in **DBiKt**. Another feature of **DBiKt** is the use

Identity and logical constants:

$$\overline{X, A \triangleright A, Y}$$
 id  $\overline{X, \bot \triangleright Y}$   $\bot_L$   $\overline{X \triangleright \top, Y}$   $\top_R$ 

Structural rules:

$$\frac{X \triangleright Z}{X, Y \triangleright Z} w_L \qquad \frac{X \triangleright Z}{X \triangleright Y, Z} w_R \qquad \frac{X, Y, Y \triangleright Z}{X, Y \triangleright Z} c_L \qquad \frac{X \triangleright Y, Y, Z}{X \triangleright Y, Z} c_R$$

$$\frac{(X_1 \triangleright Y_1), X_2 \triangleright Y_2}{X_1, X_2 \triangleright Y_1, Y_2} s_L \qquad \frac{X_1 \triangleright Y_1, (X_2 \triangleright Y_2)}{X_1, X_2 \triangleright Y_1, Y_2} s_R \qquad \frac{X_2 \triangleright Y_2, Y_1}{(X_2 \triangleright Y_2) \triangleright Y_1} \triangleright_L \qquad \frac{X_1, X_2 \triangleright Y_2}{X_1 \triangleright (X_2 \triangleright Y_2)} \triangleright_R$$

$$\frac{\bullet X \triangleright Y}{X \triangleright \bullet Y} rp_{\circ} \qquad \frac{\circ X \triangleright Y}{X \triangleright \bullet Y} rp_{\bullet} \qquad \frac{X_1 \triangleright Y_1, A \qquad A, X_2 \triangleright Y_2}{X_1, X_2 \triangleright Y_1, Y_2} cut$$

Logical rules:

$$\frac{X, B_i \triangleright Y}{X, B_1 \land B_2 \triangleright Y} \land_L i \in \{1, 2\} \qquad \frac{X \triangleright A, Y \qquad X \triangleright B, Y}{X \triangleright A \land B, Y} \land_R$$

$$\frac{X, A \triangleright Y \qquad X, B \triangleright Y}{X, A \lor B \triangleright Y} \lor_L \qquad \frac{X \triangleright B_i, Y}{X \triangleright B_1 \lor B_2, Y} \lor_R i \in \{1, 2\}$$

$$\frac{X \triangleright A, Y \qquad X, B \triangleright Y}{X, A \lor B \triangleright Y} \to_L \qquad \frac{X, A \triangleright B}{X \triangleright A \to B, Y} \to_R$$

$$\frac{A \triangleright B, Y}{X, A \multimap B \triangleright Y} \checkmark_L \qquad \frac{X \triangleright A, Y \qquad X, B \triangleright Y}{X \triangleright A \multimap B, Y} \to_R$$

$$\frac{A \triangleright X}{\Box A \triangleright \Diamond X} \Box_L \qquad \frac{X \triangleright \Diamond A}{X \triangleright \Box A} \Box_R \qquad \frac{A \triangleright X}{\Box A \triangleright \Diamond X} \blacksquare_L \qquad \frac{X \triangleright \Diamond A}{X \triangleright \blacksquare A} \blacksquare_R$$

Fig. 2. LBiKt: a shallow inference system for BiKt

 $\begin{array}{c|c} \circ A \triangleright X \\ \hline \Diamond A \triangleright X \\ \hline \end{pmatrix} \Diamond_L \qquad \begin{array}{c|c} X \triangleright A \\ \hline \Diamond X \triangleright \Diamond_A \\ \hline \end{pmatrix} \Diamond_R \qquad \begin{array}{c|c} \bullet A \triangleright X \\ \hline \bullet A \triangleright X \\ \hline \end{pmatrix} \downarrow_L \qquad \begin{array}{c|c} X \triangleright A \\ \hline \bullet X \triangleright A \\ \hline \end{array} \blacklozenge_R$ 

of polarities in defining contexts to which rules are applicable. For example, the premise of the  $\Box_{L1}$  rule denotes a negative context  $\Sigma$  which itself contains a formula A and a  $\bullet$ -structure, such that the  $\bullet$ -structure contains  $\Box A$ .

**DBiKt** achieves the goal of merging the DBiInt calculus [19] and a two-sided version of the DKt calculus [10]. While in the shallow inference case, a calculus for BiKt could be obtained relatively easily by merging shallow inference calculi for BiInt and tense logics, the combination of calculi is not so obvious in the deep inference case. Although the propagation rules for  $\triangleright$ -structures remain the same as in the BiInt case [19], the propagation rules for  $\triangleright$ - and  $\bullet$ -structures are not as simple as in the DKt calculus [10]. Since we do not assume any direct relationship between  $\square$  and  $\diamondsuit$ , or  $\blacksquare$  and  $\diamondsuit$ , propagation rules like  $\blacksquare_{L2}$  need to involve the  $\triangleright$  structural connective so they can refer to both sides of the nested sequent.

Note that in the rules  $\to_L$  and  $-\!\!\!<_R$  in  $\mathbf{DBiKt}$ , we require that the contexts in which the principal formulae reside are strict contexts. This is strictly speaking not necessary, i.e., we could remove the proviso without affecting the expressivity of the proof system. The proviso does, however, reduce the non-determinism in partitioning the contexts in  $\to_L$  or  $-\!\!\!<_R$ . Consider, for example, the nested sequent  $\circ(a,b\to c) \triangleright d$ . Without the requirement of strict contexts, there are two instances

Identity and logical constants:

$$\frac{\phantom{a}}{\Sigma[X,A\triangleright A,Y]} id \qquad \frac{\phantom{a}}{\Sigma[\bot,X\triangleright Y]} \bot_L \qquad \frac{\phantom{a}}{\Sigma[X\triangleright \top,Y]} \top_R$$

Propagation rules:

$$\frac{\Sigma^{-}[A,(A,X\triangleright Y)]}{\Sigma^{-}[A,X\triangleright Y]} \triangleright_{L1} \qquad \frac{\Sigma^{+}[X\triangleright Y,A]}{\Sigma^{+}[X\triangleright Y,A]} \triangleright_{R1}$$

$$\frac{\Sigma[X,A\triangleright W,(A,Y\triangleright Z)]}{\Sigma[X,A\triangleright W,(Y\triangleright Z)]} \triangleright_{L2} \qquad \frac{\Sigma[(X\triangleright Y,A),W\triangleright A,Z]}{\Sigma[(X\triangleright Y),W\triangleright A,Z]} \triangleright_{R2}$$

$$\frac{\Sigma^{-}[A,\bullet(\square A,X)]}{\Sigma^{-}[\bullet(\square A,X)]} \square_{L1} \qquad \frac{\Sigma^{+}[A,\bullet(\lozenge A,X)]}{\Sigma^{+}[\bullet(\lozenge A,X)]} \lozenge_{R1} \qquad \frac{\Sigma^{-}[A,\circ(\blacksquare A,X)]}{\Sigma^{-}[\circ(\blacksquare A,X)]} \blacksquare_{L1} \qquad \frac{\Sigma^{+}[A,\circ(\lozenge A,X)]}{\Sigma^{+}[\circ(\lozenge A,X)]} \lozenge_{R1}$$

$$\frac{\Sigma[\blacksquare A,X\triangleright \bullet(A\triangleright Y),Z]}{\Sigma[\blacksquare A,X\triangleright \bullet Y,Z]} \blacksquare_{L2} \qquad \frac{\Sigma[\circ(X\triangleright A),Y\triangleright Z,\lozenge A]}{\Sigma[\circ X,Y\triangleright Z,\lozenge A]} \lozenge_{R2}$$

$$\frac{\Sigma[\square A,X\triangleright \circ(A\triangleright Y),Z]}{\Sigma[\square A,X\triangleright \circ Y,Z]} \square_{L2} \qquad \frac{\Sigma[\bullet(X\triangleright A),Y\triangleright Z,\lozenge A]}{\Sigma[\bullet X,Y\triangleright Z,\lozenge A]} \lozenge_{R2}$$

Logical rules

$$\frac{\Sigma^{-}[A \vee B, A]}{\Sigma^{-}[A \vee B]} \xrightarrow{\Sigma^{-}[A \vee B, B]} \vee_{L} \qquad \frac{\Sigma^{+}[A \vee B, A, B]}{\Sigma^{+}[A \vee B]} \vee_{R}$$

$$\frac{\Sigma^{-}[A \wedge B, A, B]}{\Sigma^{-}[A \wedge B]} \wedge_{L} \qquad \frac{\Sigma^{+}[A \wedge B, A]}{\Sigma^{+}[A \wedge B]} \wedge_{R}$$

$$\frac{\Sigma^{-}[A \leftarrow B, (A \triangleright B)]}{\Sigma^{-}[A \leftarrow B]} \leftarrow_{L} \qquad \frac{\Sigma^{+}[A \rightarrow B, (A \triangleright B)]}{\Sigma^{+}[A \rightarrow B]} \rightarrow_{R}$$

$$\frac{\Sigma^{-}[X, A \rightarrow B \triangleright A]}{\Sigma^{-}[X, A \rightarrow B]} \xrightarrow{\Sigma^{-}[X, A \rightarrow B, B]} \rightarrow_{L} \qquad \Sigma^{-}[] \text{ is a strict context}$$

$$\frac{\Sigma^{+}[X, A \leftarrow B, A]}{\Sigma^{+}[X, A \leftarrow B]} \wedge_{R} \qquad \Sigma^{+}[] \text{ is a strict context}$$

$$\frac{\Sigma^{-}[\Diamond A, \circ A]}{\Sigma^{-}[\Diamond A]} \Diamond_{L} \qquad \frac{\Sigma^{+}[\Box A, \circ A]}{\Sigma^{+}[\Box A]} \Box_{R} \qquad \frac{\Sigma^{-}[\blacklozenge A, \bullet A]}{\Sigma^{-}[\blacklozenge A]} \blacklozenge_{L} \qquad \frac{\Sigma^{+}[\blacksquare A, \bullet A]}{\Sigma^{+}[\blacksquare A]} \blacksquare_{R}$$

Fig. 3. **DBiKt**: a deep inference system for BiKt

of  $\rightarrow_L$  with that nested sequent as the conclusion:

$$\frac{\circ (a, (b \to c \triangleright b)) \triangleright d \quad \circ (a, b \to c, c) \triangleright d}{\circ (a, b \to c) \triangleright d} \to_{L}$$

$$\frac{\circ (a, b \to c \triangleright b) \triangleright d \quad \circ (a, b \triangleright c, c) \triangleright d}{\circ (a, b \to c) \triangleright d} \to_{L}$$

In the first instance, the context is  $\circ(a,[\ ]) \triangleright d$ , which is not strict, whereas in the second instance, it is  $\circ([\ ]) \triangleright d$ , which is strict. In general, if there are n formulae connected to  $b \to c$  via the disjunctive structural connective, then there are  $2^n$  possible instances of  $\to_L$  without the strict context proviso.

We write  $\vdash_{\mathbf{LBiKt}} \pi : X \triangleright Y$  when  $\pi$  is a derivation of the shallow sequent  $X \triangleright Y$  in  $\mathbf{LBiKt}$ , and  $\vdash_{\mathbf{DBiKt}} \pi : X \triangleright Y$  when  $\pi$  is a derivation of the sequent  $X \triangleright Y$  in

**DBiKt**. In either calculus, the height  $|\pi|$  of a derivation  $\pi$  is the number of sequents on the longest branch.

**Example 3.1** Below we derive Ewald's axiom 9 for  $IK_t$  [6] in **LBiKt** and **DBiKt**. The **LBiKt**-derivation on the left read bottom-up brings the required sub-structure  $\blacklozenge A$  to the top-level using the residuation rule  $rp_o$  and applies  $\blacklozenge_R$  backward. The **DBiKt**-derivation on the right instead applies  $\Box_R$  deeply, and propagates the required formulae to the appropriate sub-structure using  $\blacklozenge_{R1}$ . Note that contraction is implicit in  $\blacklozenge_{R1}$ , and all propagation rules.

$$\frac{A \triangleright A}{\bullet A \triangleright A} \stackrel{id}{\bullet_R} \\
\frac{A \triangleright A \triangleright A}{A \triangleright A} \stackrel{rp_o}{\bullet_R} \\
\frac{A \triangleright A}{A \triangleright A} \stackrel{\Box}{\bullet_R} \\
\frac{A \triangleright A}{A} \stackrel$$

### Display property

A (deep or shallow) nested sequent can be seen as a tree of traditional sequents. The structural rules of **LBiKt** allows shuffling of structures to display/un-display a particular node in the tree, so inference rules can be applied to it. This is similar to the display property in traditional display calculi, where any substructure can be displayed and un-displayed. We state the display property of **LBiKt** more precisely in subsequent lemmas. We shall use two "display" rules which are easily derivable using  $s_L$ ,  $s_R$ ,  $\triangleright_L$  and  $\triangleright_R$ :

$$\frac{(X_1 \triangleright X_2) \triangleright Y}{X_1 \triangleright X_2, Y} r p_{\triangleright}^L \qquad \frac{X_1 \triangleright (X_2 \triangleright Y)}{X_1, X_2 \triangleright Y} r p_{\triangleright}^R$$

Let  $DP = \{rp_{\triangleright}^L, rp_{\triangleright}^R, rp_{\circ}, rp_{\bullet}\}$  and let DP-derivable mean "derivable using rules only from DP".

The following lemmas can be proved by simple induction on the size of the context  $\Sigma[\ ]$ .

Lemma 3.2 (Display property for neutral contexts) Let  $\Sigma[]$  be a neutral context. Let X be a structure and p a propositional variable not occurring in X nor  $\Sigma[]$ . Then there exist structures Y and Z such that:

- (i)  $Y \triangleright p$  is DP-derivable from  $X \triangleright \Sigma[p]$  and
- (ii)  $p \triangleright Z$  is DP-derivable from  $\Sigma[p] \triangleright X$ .

Lemma 3.3 (Display property for positive contexts) Let  $\Sigma[]$  be a positive context. Let X be a structure and p a propositional variable not occurring in X nor  $\Sigma[]$ . Then there exist structures Y and Z such that:

- (i)  $Y \triangleright p$  is DP-derivable from  $X \triangleright \Sigma[p]$ , and
- (ii)  $Z \triangleright p$  is DP-derivable from  $\Sigma[p] \triangleright X$ .

Lemma 3.4 (Display property for negative contexts) Let  $\Sigma[]$  be a negative context. Let X be a structure and p a propositional variable not occurring in X nor  $\Sigma[]$ . Then there exist structures Y and Z such that:

Fig. 4. An example of cut reduction

- (i)  $p \triangleright Y$  is DP-derivable from  $X \triangleright \Sigma[p]$  and
- (ii)  $p \triangleright Z$  is DP-derivable from  $\Sigma[p] \triangleright X$ .

Since the rules in DP are all invertible, the derivations constructed in the above lemmas are invertible derivations. That is, we can derive  $Y \triangleright p$  from  $X \triangleright \Sigma[p]$  and vice versa. Note also that since rules in the shallow system are closed under substitution, this also means  $Y \triangleright Z$  is derivable from  $X \triangleright \Sigma[Z]$ , and vice versa, for any Z.

The display property of pure display calculi is the ability to display/un-display a structure with respect to a top-level turnstile  $\vdash$  (say) as the *whole* of the antecedent or succedent. For example, we have to display  $V \triangleright W$  as the whole of the antecedent or succedent as  $V \triangleright W \vdash Z$  or  $Z \vdash V \triangleright W$ . Our shallow nested sequent calculus instead enables us to "zoom in" to  $V \triangleright W$  in  $X \triangleright Y$  by explicitly transforming the latter into  $X', V \triangleright W, Y'$  so a rule can be applied to any top-level formula/structure of V or W. Our deep nested sequent calculus allows us to "zoom in" to  $V \triangleright W$  by treating it as the filler of a hole  $\Sigma[V \triangleright W]$ , without explicit transformations.

### 4 Cut elimination in LBiKt

Our cut-elimination proof is based on the method of proof-substitution presented in [9]. It is very similar to the general cut elimination method used in display calculi. The proof relies on the display property and the fact that inference rules in **LBiKt** are closed under substitutions.

We illustrate the cut reduction steps here with an example. Consider the derivation below ending with a cut on  $\Diamond A$ :

$$\frac{X_1 \triangleright Y_1, \lozenge A \qquad \lozenge A, X_2 \triangleright Y_2}{X_1, X_2 \triangleright Y_1, Y_2} cut$$

Instead of permuting the cut rule locally, we trace the cut formula  $\Diamond A$  until it becomes principal in the derivations  $\pi_1$  and  $\pi_2$ , and then apply cut on a smaller formula. Suppose that  $\pi_1$  and  $\pi_2$  are respectively the derivations (1) and (2) in Figure 4. We first transform  $\pi_1$  by substituting  $(X_2 \triangleright Y_2)$  for  $\Diamond A$  in  $\pi_1$  and obtain the sub-derivation with an open leaf as shown in Figure 4(3). We then prove the

open leaf by uniformly substituting  $\circ(X_1')$  for  $\Diamond A$  in  $\pi_2$ , and applying cut on a sub-formula A, as shown in Figure 4(4).

The cut rank of an instance of cut is the size of the cut formula, as usual. The cut rank  $cr(\pi)$  of a derivation  $\pi$  is the largest cut rank of the cut instances in  $\pi$  (or zero, if  $\pi$  is cut-free). Given a formula A, we denote with |A| its size.

To formalise the cut elimination proof, we first introduce a notion of multiple-hole contexts. A k-hole context is a context with k holes. Given a k-hole context  $\Sigma[\cdots]$  we write  $\Sigma[X^k]$  to stand for the structure obtained from  $\Sigma[\cdots]$  by replacing each hole with an occurrence of the structure X. For example, if  $\Sigma[] = ([], \bullet([], W \triangleright Y)) \triangleright Z$  then  $\Sigma[(\circ U)^2] = (\circ U, \bullet(\circ U, W \triangleright Y)) \triangleright Z$ . A neutral (resp. positive and negative) k-hole context is a k-hole context where all the holes have neutral (resp. positive and negative) polarity. A quasi-positive (resp. quasi-negative) k-hole context is a k-hole context where each hole in the context has either neutral or positive (resp. negative) polarity. Obviously, a k-hole positive (negative) context is also a k-hole quasi-positive (quasi-negative) context.

Lemma 4.1 states the proof substitutions needed to eliminate atomic cuts. Lemmas 4.2-4.9 state the proof substitutions needed for non-atomic cuts. We only give the proofs of the cases involving the modal connectives as the other proofs are unchanged from [9].

**Lemma 4.1** Suppose  $p, X \triangleright Y$  is cut-free derivable for some fixed p, X and Y. Then for any k-hole positive context  $Z_1[\cdots]$  and any l-hole quasi-positive context  $Z_2[\cdots]$ , if  $Z_1[p^k] \triangleright Z_2[p^l]$  is cut-free derivable, then  $Z_1[(X \triangleright Y)^k] \triangleright Z_2[(X \triangleright Y)^l]$  is cut-free derivable.

**Proof.** Let  $\pi_1$  be a cut-free derivation of  $p, X \triangleright Y$  and let  $\pi_2$  be a cut-free derivation of  $Z_1[p^k] \triangleright Z_2[p^l]$ . We construct a cut-free derivation  $\pi$  of  $Z_1[(X \triangleright Y)^k] \triangleright Z_2[(X \triangleright Y)^l]$  by induction on  $|\pi_2|$ . Most cases follow straightforwardly from the induction hypothesis. The only non-trivial case is when p is active in the derivation, i.e., when  $\pi_2$  ends with an id rule or a contraction rule applied to an occurrence of p to be substituted for:

\* Suppose  $\pi_2$  is

$$\overline{Z'_1[p^k], p \triangleright p, Z'_2[p^{l-1}]} id$$

Note that the p immediately to the left of the " $\triangleright$ " cannot be part of the  $p^k$  by the restrictions on the context  $Z_1[\cdots]$ . The derivation  $\pi$  is then constructed as follows, where we use double lines to abbreviate derivations:

$$\frac{p, X \triangleright Y}{p \triangleright (X \triangleright Y)} \triangleright_{R}$$

$$Z'_{1}[(X \triangleright Y)^{k}], p \triangleright (X \triangleright Y), Z'_{2}[(X \triangleright Y)^{l-1}] \quad w_{R}; w_{L}$$

\* Suppose  $\pi_2$  is

$$\frac{Z_1[p^k] \vdash p, p, Z_2'[p^{l-1}]}{Z_1[p^k] \vdash p, Z_2'[p^{l-1}]} c_R$$

By induction hypothesis, we have a cut-free derivation  $\psi'$  of

$$Z_1[(X \triangleright Y)^k] \triangleright (X \triangleright Y), (X \triangleright Y), Z_2'[(X \triangleright Y)^{l-1}].$$

The derivation  $\pi$  is then constructed as follows:

$$\frac{\psi'}{Z_1[(X \triangleright Y)^k] \triangleright (X \triangleright Y), (X \triangleright Y), Z_2'[(X \triangleright Y)^{l-1}]} c_R$$

$$\frac{Z_1[(X \triangleright Y)^k] \triangleright (X \triangleright Y), Z_2'[(X \triangleright Y)^{l-1}]}{Z_1[(X \triangleright Y)^k] \triangleright (X \triangleright Y), Z_2'[(X \triangleright Y)^{l-1}]} c_R$$

 $\dashv$ 

**Lemma 4.2** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_i : X \triangleright Y, A_i$ , for some  $i \in \{1, 2\}$ , such that  $cr(\pi_i) < |A_1 \lor A_2|$ . Suppose  $\vdash_{\mathbf{LBiKt}} \pi_3 : Z_1[(A_1 \lor A_2)^k] \triangleright Z_2[(A_1 \lor A_2)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$ , such that  $cr(\pi_3) < |A_1 \lor A_2|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[(X \triangleright Y)^k] \triangleright Z_2[(X \triangleright Y)^l]$  and  $cr(\pi) < |A \lor B|$ .

**Proof.** By induction on  $|\pi_3|$ . In the following, we let  $A = A_1 \vee A_2$ . Most cases follow straightforwardly from the induction hypothesis. The only interesting case is when a left-rule is applied to an occurrence of  $A_1 \vee A_2$  which is to be replaced by  $X \triangleright Y$ . That is,  $\pi_3$  is

$$\frac{\psi_1}{Z_1'[A^{k-1}], A_1 \triangleright Z_2[A^l]} \frac{\psi_2}{Z_1'[A^{k-1}], A_2 \triangleright Z_2[A^l]} \vee_L$$

$$\frac{Z_1'[A^{k-1}], A_1 \vee A_2 \triangleright Z_2[A^l]}{Z_1'[A^{k-1}], A_1 \vee A_2 \triangleright Z_2[A^l]} \vee_L$$

By induction hypothesis, we have a derivation  $\psi'_i$ , for each  $i \in \{1, 2\}$ , of

$$Z_1'[(X \triangleright Y)^{k-1}], A_i \triangleright Z_2[(X \triangleright Y)^l]$$

with  $cr(\psi_i') < |A_1 \vee A_2|$ . The derivation  $\pi$  is then constructed as follows:

$$\frac{X \triangleright Y, A_i}{X \triangleright Y \triangleright A_i} \triangleright_L \qquad \psi_i'$$

$$\frac{Z_1'[(X \triangleright Y)^{k-1}], A_i \triangleright Z_2[(X \triangleright Y)^l]}{Z_1'[(X \triangleright Y)^{k-1}], X \triangleright Y \triangleright Z_2[(X \triangleright Y)^l]} cut$$

 $\dashv$ 

**Lemma 4.3** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X \triangleright Y, A_1 \text{ and } \vdash_{\mathbf{LBiKt}} \pi_2 : X \triangleright Y, A_2 \text{ with } cr(\pi_1) < |A_1 \wedge A_2| \text{ and } cr(\pi_2) < |A_1 \wedge A_2|.$  Suppose  $\vdash_{\mathbf{LBiKt}} \pi_3 : Z_1[(A_1 \wedge A_2)^k] \triangleright Z_2[(A_1 \wedge A_2)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$  with  $cr(\pi_3) < |A_1 \wedge A_2|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[(X \triangleright Y)^k] \triangleright Z_2[(X \triangleright Y)^l]$  and  $cr(\pi) < |A \wedge B|$ .

**Proof.** By induction on  $|\pi_3|$ . In the following, we let  $A = A_1 \wedge A_2$ . Most cases follow straightforwardly from the induction hypothesis. The only interesting case is when a left-rule is applied to an occurrence of  $A_1 \wedge A_2$  which is to be replaced by

 $X \triangleright Y$ . That is,  $\pi_3$  is:

$$\frac{\psi_i}{Z_1[A^{k-1}], A_i \triangleright Z_2[A^l]} \wedge_L$$

$$\frac{Z_1[A^{k-1}], A_1 \wedge A_2 \triangleright Z_2[A^l]}{Z_1[A^{k-1}], A_1 \wedge A_2 \triangleright Z_2[A^l]} \wedge_L$$

for some  $i \in \{1, 2\}$ . By induction hypothesis, we have a derivation  $\psi'_i$ , for some  $i \in \{1, 2\}$ , of

$$Z_1[(X \triangleright Y)^{k-1}], A_i \triangleright Z_2[(X \triangleright Y)^l]$$

with  $cr(\psi_i) < |A_1 \wedge A_2|$ . The derivation  $\pi$  is then constructed as follows:

$$\frac{X \triangleright Y, A_i}{X \triangleright Y \triangleright A_i} \triangleright_L \qquad \psi_i'$$

$$\frac{X \triangleright Y \triangleright A_i}{Z_1'[(X \triangleright Y)^{k-1}], A_i \triangleright Z_2[(X \triangleright Y)^l]} cut$$

 $\dashv$ 

**Lemma 4.4** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X, A \triangleright B$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : Z_1[(A \rightarrow B)^k] \triangleright Z_2[(A \rightarrow B)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$ , and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than  $|A \rightarrow B|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[X^k] \triangleright Z_2[X^l]$  and  $cr(\pi) < |A \rightarrow B|$ .

**Proof.** By induction on  $|\pi_2|$ . The non-trivial case is when  $\pi_2$  ends with  $\to_L$  on  $A \to B$ :

$$\frac{U_1}{Z_1'[(A \to B)^{k-1}] \triangleright A, Z_2[(A \to B)^l]} \frac{V_2}{Z_1'[(A \to B)^{k-1}], B \triangleright Z_2[(A \to B)^l]} \to_L Z_1'[(A \to B)^{k-1}], A \to B \triangleright Z_2[(A \to B)^l]$$

By induction hypothesis, we have derivations  $\psi'_1$  and  $\psi'_2$  respectively of the sequents below where  $cr(\psi'_1) < |A \to B|$  and  $cr(\psi'_2) < |A \to B|$ :

$$Z'_1[X^{k-1}] \triangleright A, Z_2[X^l]$$
  $Z'_1[X^{k-1}], B \triangleright Z_2[X^l]$ 

In the following, we let  $V_1$  denote  $Z'_1[X^{k-1}]$  and  $V_2$  denote  $Z_2[X^l]$ . The derivation  $\pi$  is constructed as follows:

$$\frac{\psi_{1}' \qquad \frac{x, A \triangleright B \qquad V_{1}, B \triangleright V_{2}}{X, A \triangleright B \qquad V_{1}, A, X \triangleright V_{2}} cut} \underbrace{\frac{V_{1}, V_{1}, X \triangleright V_{2}, V_{2}}{V_{1}, X \triangleright V_{2}} cut}_{T_{1}, X \triangleright V_{2}} cut$$

 $\dashv$ 

**Lemma 4.5** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X \rhd Y, A$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : X, B \rhd Y,$  and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than  $|A \multimap B|$ . Suppose  $\vdash_{\mathbf{LBiKt}} \pi_3 : Z_1[(A \multimap B)^k] \rhd Z_2[(A \multimap B)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$  with  $cr(\pi_3) < |A \multimap B|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[(X \rhd Y)^k] \rhd Z_2[(X \rhd Y)^l]$  and  $cr(\pi) < |A \multimap B|$ .

By induction on  $|\pi_2|$ . The non-trivial case is when  $\pi_2$  ends with  $\prec L$  on  $A \prec B$ :

$$\frac{\psi}{Z_1'[(A \prec B)^{k-1}], A \prec B \triangleright Z_2[(A \prec B)^l]} \prec L$$

By induction hypothesis, we have a derivation  $\psi'$  of

$$A \triangleright B, Z_2[(X \triangleright Y)^l]$$

with  $cr(\psi) < |A| - \langle B|$ . The derivation  $\pi$  is then constructed as follows:

$$\frac{\pi_{1}}{X \triangleright Y, A} \frac{A \triangleright B, Z_{2}[(X \triangleright Y)^{l}] \qquad X, B \triangleright Y}{A, X \triangleright Y, Z_{2}[(X \triangleright Y)^{l}]} cut$$

$$\frac{X, X \triangleright Y, Y, Z_{2}[(X \triangleright Y)^{l}]}{X \triangleright Y, Z_{2}[(X \triangleright Y)^{l}]} c_{L}; c_{R}$$

$$\frac{X \triangleright Y, Z_{2}[(X \triangleright Y)^{l}]}{Z_{1}[(X \triangleright Y)^{k-1}], X \triangleright Y \triangleright Z_{2}[(X \triangleright Y)^{l}]} \triangleright_{L}$$

**Lemma 4.6** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X \rhd \circ A$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : Z_1[(\Box A)^k] \rhd Z_2[(\Box A)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$ , and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than  $|\Box A|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[X^k] \rhd Z_2[X^l]$  and  $cr(\pi) < |\Box A|$ .

**Proof.** By induction on  $|\pi_2|$ . The non-trivial case is when  $\pi_2$  ends with  $\Box_L$  on  $\Box A$  as shown below left. By induction hypothesis we have  $\vdash_{\mathbf{LBiKt}} \psi' : A \triangleright Z_2'[X^l]$  where  $cr(\psi') < |\Box A|$ . The derivation  $\pi$  is constructed as shown below right:

$$\frac{\psi}{A \triangleright Z_{2}'[(\Box A)^{l}]} \Box_{L} \qquad \frac{X \triangleright A}{\bullet X \triangleright A} rp_{\circ} \qquad \psi' \\
\frac{A \triangleright Z_{2}'[(\Box A)^{l}]}{A \triangleright (Z_{2}'[(\Box A)^{l}])} \Box_{L} \qquad \frac{\bullet X \triangleright Z_{2}'[X^{l}]}{X \triangleright (Z_{2}'[X^{l}])} rp_{\circ}$$

 $\dashv$ 

**Lemma 4.7** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X \triangleright \bullet A$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : Z_1[(\blacksquare A)^k] \triangleright Z_2[(\blacksquare A)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$ , and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than  $|\blacksquare A|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[X^k] \triangleright Z_2[X^l]$  and  $cr(\pi) < |\blacksquare A|$ .

**Proof.** By induction on  $|\pi_2|$ . The non-trivial case is when  $\pi_2$  ends with  $\blacksquare_L$  on  $\blacksquare A$  as shown below left. By induction hypothesis we have  $\vdash_{\mathbf{LBiKt}} \psi' : A \triangleright Z_2'[X^l]$  where

 $cr(\psi') < |\blacksquare A|$ . The derivation  $\pi$  is constructed as shown below right:

$$\frac{A \triangleright Z_2'[(\blacksquare A)^l]}{\blacksquare A \triangleright \bullet (Z_2'[(\blacksquare A)^l])} \blacksquare_L \qquad \frac{X \triangleright \bullet A}{\circ X \triangleright A} rp_{\bullet} \qquad \psi' \\
\frac{A \triangleright Z_2'[X^l]}{A \triangleright Z_2'[X^l]} cut$$

$$\frac{(A \triangleright Z_2'[X^l])}{A \triangleright Z_2'[X^l]} rp_{\bullet}$$

 $\dashv$ 

**Lemma 4.8** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X \triangleright A$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : Z_1[(\lozenge A)^k] \triangleright Z_2[(\lozenge A)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$ , and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than  $|\lozenge A|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[(\lozenge X)^k] \triangleright Z_2[(\lozenge X)^l]$  and  $cr(\pi) < |\lozenge A|$ .

**Proof.** By induction on  $|\pi_2|$ . The non-trivial case is when  $\pi_2$  ends with  $\Diamond_L$  on  $\Diamond A$  as shown below left. By induction hypothesis we have  $\vdash_{\mathbf{LBiKt}} \psi' : \circ A \rhd Z_2[(\circ X)^l]$  where  $cr(\psi') < |\Diamond A|$ . The derivation  $\pi$  is constructed as shown below right:

$$\frac{\psi}{ \begin{array}{c} \Diamond A \triangleright Z_{2}[(\Diamond A)^{l}] \\ \Diamond A \triangleright Z_{2}[(\Diamond A)^{l}] \end{array}} \Diamond_{L} \qquad \frac{\pi_{1}}{ \begin{array}{c} (A \triangleright Z_{2}[(\circ X)^{l}]) \\ X \triangleright A \end{array}} \frac{\neg A \triangleright Z_{2}[(\circ X)^{l}]}{ \begin{array}{c} (A \triangleright Z_{2}[(\circ X)^{l}]) \\ \hline (A \triangleright Z_{2}[(\circ X)^{l}]) \end{array}} rp_{\bullet} cut$$

 $\dashv$ 

**Lemma 4.9** Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : X \triangleright A$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : Z_1[(\blacklozenge A)^k] \triangleright Z_2[(\blacklozenge A)^l]$  for some k-hole quasi-negative context  $Z_1[\cdots]$  and l-hole negative context  $Z_2[\cdots]$ , and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than  $|\blacklozenge A|$ . Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[(\bullet X)^k] \triangleright Z_2[(\bullet X)^l]$  and  $cr(\pi) < |\blacklozenge A|$ .

**Proof.** By induction on  $|\pi_2|$ . The non-trivial case is when  $\pi_2$  ends with  $\blacklozenge_L$  on  $\blacklozenge A$  as shown below left. By induction hypothesis we have  $\vdash_{\mathbf{LBiKt}} \psi' : \bullet A \rhd Z_2[(\bullet X)^l]$  where  $cr(\psi') < |\blacklozenge A|$ . The derivation  $\pi$  is constructed as shown below right:

$$\psi$$

$$\bullet A \triangleright Z_{2}[(\blacklozenge A)^{l}]$$

$$\bullet Z_{2}[(\blacklozenge A)^{l}]$$

$$\bullet Z_{2}[(\blacklozenge X)^{l}]$$

 $\dashv$ 

**Lemma 4.10** Let A be a non-atomic formula. Suppose  $\vdash_{\mathbf{LBiKt}} \pi_1 : A, X \triangleright Y$  and  $\vdash_{\mathbf{LBiKt}} \pi_2 : Z_1[A^k] \triangleright Z_2[A^l]$  where  $Z_1[\cdots]$  is a k-hole positive context,  $Z_2[\cdots]$  is an l-hole quasi-positive context, and the cut ranks of  $\pi_1$  and  $\pi_2$  are smaller than |A|. Then there exists  $\pi$  such that  $\vdash_{\mathbf{LBiKt}} \pi : Z_1[(X\triangleright Y)^k] \triangleright Z_2[(X\triangleright Y)^l]$  and  $cr(\pi) < |A|$ .

**Proof.** By induction on  $|\pi_2|$  and case analysis on A. The non-trivial case is when  $\pi_2$  ends with a right-introduction rule on A. That is, in this case, we have  $Z_2[A^l] =$ 

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 $(Z_2'[A^{l-1}], A)$  for some quasi-positive context  $Z_2'[\cdots]$ . We distinguish several cases depending on A. We show here the cases where A is either  $\Box C$  or  $\Diamond C$ .

\* Suppose  $A = \Box C$  and  $\pi_2$  is as below left. By induction hypothesis, we have a derivation  $\psi'$  of  $Z_1[(X \triangleright Y)^k] \triangleright \circ C$ . Then the derivation  $\pi$  is constructed as shown below right, with  $\theta$  obtained by applying Lemma 4.6 to  $\psi'$  and  $\pi_1$ .

$$\frac{\psi}{Z_1[(\Box C)^k] \triangleright \circ C} \Box_R \qquad \frac{Z_1[(X \triangleright Y)^k], X \triangleright Y}{Z_1[(X \triangleright Y)^k] \triangleright X \triangleright Y} \triangleright_R$$

\* Suppose  $A = \Diamond C$  and  $\pi_2$  is as below left. By induction hypothesis, we have a derivation  $\psi'$  of  $Z'_1[(X \triangleright Y)^k] \triangleright C$ . Then the derivation  $\pi$  is constructed as shown below right, with  $\theta$  obtained by applying Lemma 4.8 to  $\psi'$  and  $\pi_1$ .

$$\frac{\psi}{Z_1'[(\lozenge C)^k] \triangleright C} \lozenge_R \qquad \frac{\theta}{\circ (Z_1'[(\lozenge C)^k]) \triangleright \lozenge C} \lozenge_R \qquad \frac{\circ (Z_1'[(X \triangleright Y)^k]), X \triangleright Y}{\circ (Z_1'[(X \triangleright Y)^k]) \triangleright X \triangleright Y} \triangleright_R$$

The other cases are treated analogously, using Lemmas 4.2, 4.3, 4.4, 4.5, 4.7 and 4.9.  $\dashv$ 

Theorem 4.11 (Cut elimination for LBiKt) If  $X \triangleright Y$  is LBiKt-derivable then it is also LBiKt-derivable without using cut.

**Proof.** As typical in cut elimination proofs, we remove topmost cuts in succession. Let  $\pi$  be a derivation of **LBiKt** with a topmost cut instance

$$\frac{X_1 \triangleright Y_1, A \qquad X_2 \triangleright Y_2}{X_1, X_2 \triangleright Y_1, Y_2} \text{ cut}$$

Note that  $\pi_1$  and  $\pi_2$  are both cut-free since this is a topmost instance in  $\pi$ . We use induction on the size of A to eliminate this topmost instance of cut.

If A is an atomic formula p then the cut free derivation is constructed as follows where  $\psi$  is obtained from applying Lemma 4.1 to  $\pi_2$  and  $\pi_1$ :

$$\frac{\psi}{X_1 \triangleright Y_1, (X_2 \triangleright Y_2)} s_R$$

$$\frac{X_1, X_2 \triangleright Y_1, Y_2}{X_1, X_2 \triangleright Y_1, Y_2} s_R$$

If A is non-atomic, using Lemma 4.10 we get the following derivation  $\pi'$ :

$$\frac{\psi}{X_1 \triangleright Y_1, (X_2 \triangleright Y_2)} s_R$$

We have  $cr(\pi') < |A|$  by Lemma 4.10, therefore by induction hypothesis, we can remove all the cuts in  $\pi'$  to get a cut-free derivation of  $X_1, X_2 \triangleright Y_1, Y_2$ .  $\dashv$ 

# 5 Equivalence between LBiKt and DBiKt

We now show that **LBiKt** and **DBiKt** are equivalent. We first show that every derivation in **DBiKt** can be mimicked by a cut-free derivation in **LBiKt**. The interesting cases involve showing that the propagation rules of **DBiKt** are derivable in **LBiKt** using residuation. This is not surprising since the residuation rules in display calculi are used exactly for the purpose of displaying and un-displaying sub-structures so that inference rules can be applied to them.

**Theorem 5.1** For any 
$$X$$
 and  $Y$ , if  $\vdash_{\mathbf{DBiKt}} \pi : X \triangleright Y$  then  $\vdash_{\mathbf{LBiKt}} \pi' : X \triangleright Y$ .

**Proof.** We show that each deep inference rule  $\rho$  is derivable in the shallow system. This is done by case analysis of the context  $\Sigma[\ ]$  in which the deep rule  $\rho$  applies. Note that if a deep inference rule  $\rho$  is applicable to  $X \triangleright Y$ , then the context  $\Sigma[\ ]$  in this case is either  $[\ ]$ , a positive context or a negative context. In the first case, it is easy to show that each valid instance of  $\rho$  where  $\Sigma[\ ] = [\ ]$  is derivable in the shallow system. For the case where  $\Sigma[\ ]$  is either positive or negative, we use the display property. We show here the case where  $\rho$  is a rule with a single premise; the other cases are similar. Suppose  $\rho$  is as below left. By the display properties, we need only to show that the rule shown below right is derivable in the shallow system for some structure X':

$$\frac{\Sigma^{+}[U]}{\Sigma^{+}[V]} \rho \qquad \frac{X' \triangleright U}{X' \triangleright V}$$

For example, to show soundness of  $\square_{L2}$  it is enough to show that the following are derivable:

$$\frac{R \triangleright (\Box A, X \triangleright \circ (A \triangleright Y), Z)}{R \triangleright (\Box A, X \triangleright \circ Y, Z)} \qquad \frac{(\Box A, X \triangleright \circ (A \triangleright Y), Z) \triangleright R}{(\Box A, X \triangleright \circ Y, Z) \triangleright R}$$

Both reduce to showing that the following rule shown below left is derivable; the derivation below right gives the required:

$$\begin{array}{c} \square A, X \rhd \circ (A \rhd Y), Z \\ \hline \square A, X \rhd Z \rhd \circ (A \rhd Y) \\ \hline \bullet (\square A, X \rhd Z) \rhd (A \rhd Y) \\ \hline A, \bullet (\square A, X \rhd Z) \rhd Y \\ \hline A \rhd \bullet (\square A, X \rhd Z) \rhd Y \\ \hline \square A, X \rhd \circ (A \rhd Y), Z \\ \hline \square A, X \rhd \circ (A \rhd Y), Z \\ \hline \hline \square A, X \rhd \circ (A \rhd Y), Z \\ \hline \hline \square A, X \rhd (\square A, X \rhd Z) \rhd Y \\ \hline \hline \square A, X \rhd (\square A, X \rhd Z) \rhd Y \\ \hline \bullet (\square A, X \rhd Z) \rhd \bullet (\square A, X \rhd Z) \rhd Y \\ \hline \bullet (\square A, X \rhd Z) \rhd \bullet (\square A, X \rhd Z) \rhd Y \\ \hline \bullet (\square A, X \rhd Z), \bullet (\square A, X \rhd Z) \rhd Y \\ \hline \hline \bullet (\square A, X \rhd Z), \bullet (\square A, X \rhd Z) \rhd Y \\ \hline \hline \bullet (\square A, X \rhd Z), \circ (\square A, X \rhd Z), \circ Y \\ \hline \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma (\square A, X \rhd Z), \sigma Y \\ \hline \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma (\square A, X \rhd Z), \sigma Y \\ \hline \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma (\square A, X \rhd Z), \sigma Y \\ \hline \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma (\square A, X \rhd Z), \sigma Y \\ \hline \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma (\square A, X \rhd Z), \sigma Y \\ \hline \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, X \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, Z \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, Z \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, Z \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, Z \rhd (\square A, X \rhd Z), \sigma Y \\ \hline \Box A, Z \rhd (\square A, X \rhd Z), \sigma Y \\ \Box$$

Below are some other cases, the rest are similar or easier:

$$\frac{Z \triangleright (X \triangleright Y, A), A, W}{Z \triangleright (X \triangleright Y, A), A, W} \triangleright_{R1} \qquad \sim \qquad \frac{\frac{Z \triangleright (X \triangleright Y, A), A, W}{(Z \triangleright W) \triangleright (X \triangleright Y, A), A} \triangleright_{R}}{\frac{(Z \triangleright W), X \triangleright Y, A, A}{(Z \triangleright W), X \triangleright Y, A} \triangleright_{R}}{\frac{(Z \triangleright W), X \triangleright Y, A, W}{Z \triangleright (X \triangleright Y, A), W}} \triangleright_{R} \\ \frac{Z, A, \circ (\blacksquare A, X) \triangleright Y}{Z, \circ (\blacksquare A, X) \triangleright Y} \blacksquare_{L1} \qquad \sim \qquad \frac{Z, A, \circ (\blacksquare A, X) \triangleright (Z \triangleright Y)}{\blacksquare_{A, X} \triangleright (\circ (\blacksquare A, X) \triangleright (Z \triangleright Y))} \blacksquare_{L} \\ \frac{Z, A, \circ (\blacksquare A, X) \triangleright Y}{\blacksquare_{A, X} \triangleright (\circ (\blacksquare A, X) \triangleright (Z \triangleright Y))} \blacksquare_{L} \\ \frac{Z, A, \circ (\blacksquare A, X) \triangleright Y}{\blacksquare_{A, X} \triangleright (\circ (\blacksquare A, X) \triangleright (Z \triangleright Y))} \blacksquare_{L} \\ \frac{Z, A, \circ (\blacksquare A, X) \triangleright (\circ (\blacksquare A, X) \triangleright (Z \triangleright Y))}{\bullet (\blacksquare A, X) \triangleright (\circ (\blacksquare A, X) \triangleright (Z \triangleright Y))} rp_{\bullet}$$

 $\frac{\blacksquare A \rhd \bullet (\circ (\blacksquare A, X) \rhd (Z \rhd Y))}{\blacksquare A, X \rhd \bullet (\circ (\blacksquare A, X) \rhd (Z \rhd Y))} \xrightarrow{w_L} w_L$   $\frac{\circ (\blacksquare A, X) \rhd (\circ (\blacksquare A, X) \rhd (Z \rhd Y))}{\circ (\blacksquare A, X), \circ (\blacksquare A, X) \rhd (Z \rhd Y)} \xrightarrow{s_R} c_L$   $\frac{\circ (\blacksquare A, X) \rhd (Z \rhd Y)}{Z, \circ (\blacksquare A, X) \rhd Y} \xrightarrow{s_R} c_L$ 

 $\dashv$ 

We now show that any *cut-free* **LBiKt**-derivation can be transformed into a cut-free **DBiKt**-derivation. This requires proving cut-free admissibility of various structural rules in **DBiKt**. The admissibility of general weakening and *formula* contraction (not general contraction, which we will show later) are straightforward by induction on the height of derivations.

Lemma 5.2 (Admissibility of general weakening) For any structures X and  $Y: if \vdash_{\mathbf{DBiKt}} \pi : \Sigma[X]$  then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[X,Y]$  such that  $|\pi'| \leq |\pi|$ .

**Lemma 5.3 (Admissibility of formula contraction)** For any structure X and formula A: if  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma[X, A, A]$  then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[X, A]$  such that  $|\pi'| \leq |\pi|$ .

Once weakening and formula contraction are shown admissible, it remains to show that the residuation rules of  $\mathbf{LBiKt}$  are also admissible. In contrast to the case with the deep inference system for bi-intuitionistic logic, the combination of modal and intuitionistic structural connectives complicates the proof of this admissibility. It seems crucial to first show "deep" admissibility of certain forms of residuation for  $\triangleright$ . We state the required lemmas below.

Unless stated otherwise, all lemmas in this section are proved by induction on  $|\pi|$ , and  $\pi'_1$  is obtained from  $\pi_1$  using the induction hypothesis. We label a dashed line with the lemma used to obtain the conclusion from the premise.

Lemma 5.4 (Deep admissibility of structural rules) The following statements hold for DBiKt:

(i) Deep admissibility of  $s_L$ . If  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma[(X \triangleright Y), Z \triangleright W]$  then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[X, Z \triangleright Y, W]$  such that  $|\pi'| \leq |\pi|$ .

- (ii) Deep admissibility of  $s_R$ . If  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma[X \triangleright Y, (Z \triangleright W)]$  then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[X, Z \triangleright Y, W]$  such that  $|\pi'| \leq |\pi|$ .
- (iii) Deep admissibility of  $\triangleright_L$ . If  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma[X \triangleright Y, Z]$  and  $\Sigma$  is either the empty context  $[\ ]$  or a negative context  $\Sigma_1^-[\ ]$ , then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[(X \triangleright Y) \triangleright Z]$ .
- (iv) Deep admissibility of  $\triangleright_R$ . If  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma[X, Y \triangleright Z]$  and  $\Sigma$  is either the empty context  $[\ ]$  or a positive context  $\Sigma_1^+[\ ]$ , then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[X \triangleright (Y \triangleright Z)]$ .

**Proof.** We prove item (i) and item (iv); the other two items can be proved symmetrically. Both are proved by induction on the height of  $\pi$ .

(i): The only interesting cases to consider are ones in which  $\pi$  ends with a propagation rule that moves a formula into or out of the structure  $(X \triangleright Y)$ , or, one which moves a formula across from X to Y or vice versa. We show here one case for each of these movements, involving the propagation rule  $\triangleright_{L1}$  and  $\triangleright_{L2}$ . For simplicity, we omit the context  $\Sigma[\ ]$  as it is not affected by the propagation rule.

Suppose  $(X \triangleright Y) = (A, X' \triangleright Y)$  and  $\pi$  is as given below left. Then  $\pi$  is transformed to  $\pi'$  (below right). Note that since formula contraction is height-preserving admissible, we have that  $|\pi'| \le |\pi'_1| \le |\pi_1| < |\pi|$ .

$$\frac{\pi_1}{(A, X' \triangleright Y), A, Z \triangleright W} \triangleright_{L1} \qquad \sim \qquad \frac{\pi'_1}{A, X', A, Z \triangleright Y, W} - \text{Lm. 5.3}$$

Suppose  $(X \triangleright Y) = (X', A \triangleright (Y_1 \triangleright Y_2), Y_3)$  and  $\pi$  is as given below left. Then  $\pi'$  is given below right.

$$\frac{(X',A\rhd(A,Y_1\rhd Y_2),Y_3),Z\rhd W}{(X',A\rhd(Y_1\rhd Y_2),Y_3),Z\rhd W}\rhd_{L2} \qquad \rightsquigarrow \qquad \frac{X',A,Z\rhd(A,Z,Y_1\rhd Y_2),W}{X',A,Z\rhd(Y_1\rhd Y_2),Y_3,W} \rhd_{L2}$$

(iv): The only non-trivial cases are when  $\pi$  ends with a propagation rule that moves a formula across the context  $\Sigma[\ ]$  and the structure  $(X,Y\triangleright Z)$ ; or across (X,Y) and Z. We show here one interesting proof transformation involving the latter: Suppose  $(X,Y\triangleright Z)=(X',\blacksquare A,Y\triangleright Z_1,\bullet Z_2)$  and  $\pi$  is as given below left. Then  $\pi'$  is as given below right.

$$\frac{X', \blacksquare A, Y \triangleright Z_1, \bullet (A \triangleright Z_2)}{X', \blacksquare A, Y \triangleright Z_1, \bullet Z_2} \blacksquare_{L2} \qquad \stackrel{\pi'_1}{\longrightarrow} \frac{X', \blacksquare A \triangleright (Y \triangleright Z_1, \bullet (A \triangleright Z_2))}{X', \blacksquare A \triangleright (\blacksquare A, Y \triangleright Z_1, \bullet (A \triangleright Z_2))} \stackrel{\text{Lm. 5.2}}{\blacksquare_{L2}} \frac{X', \blacksquare A \triangleright (\blacksquare A, Y \triangleright Z_1, \bullet Z_2)}{X', \blacksquare A \triangleright (Y \triangleright Z_1, \bullet Z_2)} \triangleright_{L2}$$

 $\dashv$ 

We now show that the residuation rules of **LBiKt** for o- and •-structures are admissible in **DBiKt**, i.e., they can be simulated by the propagation rules of **DBiKt**. First we prove a more general admissibility of residuation, which is needed for the induction hypothesis of the specific cases.

Lemma 5.5 If 
$$\vdash_{\mathbf{DBiKt}} \pi : X \triangleright Y, \bullet Z \ then \vdash_{\mathbf{DBiKt}} \pi' : \circ (X \triangleright Y) \triangleright Z.$$

**Proof.** The proof is by induction on  $|\pi|$ . If  $\pi$  ends with a deep inference rule acting on a substructure inside X, Y or Z, then  $\pi'$  can be constructed straightforwardly from the induction hypothesis. We look at the more interesting cases where  $\pi$  ends with a propagation rule that moves a formula across X, Y or Z. We give here a representative case; it is not difficult to prove the other cases. Suppose  $X = (X', \blacksquare A)$  and  $\pi$  is the derivation below left. Then  $\pi'$  is constructed as shown below right, where  $\pi'_1$  is obtained by applying the induction hypothesis to  $\pi_1$ .

$$\begin{array}{c}
\pi_{1} \\
X', \blacksquare A \triangleright Y, \bullet(A \triangleright Z) \\
X', \blacksquare A \triangleright Y, \bullet(Z)
\end{array}$$

$$\xrightarrow{L2}$$

$$\begin{array}{c}
\pi'_{1} \\
\bullet(X', \blacksquare A \triangleright Y) \triangleright (A \triangleright Z) \\
A, \circ(X', \blacksquare A \triangleright Y) \triangleright Z \\
\hline
\bullet(X', \blacksquare A \triangleright Y) \triangleright Z
\end{array}$$

$$\begin{array}{c}
Lm \ 5.4(ii) \\
A, \circ(\blacksquare A, (X', \blacksquare A \triangleright Y)) \triangleright Z \\
\hline
\bullet(X', \blacksquare A \triangleright Y) \triangleright Z
\end{array}$$

$$\begin{array}{c}
\mu_{1} \\
\bullet(X', \blacksquare A \triangleright Y) \triangleright Z
\end{array}$$

$$\begin{array}{c}
(\blacksquare A, (X', \blacksquare A \triangleright Y)) \triangleright Z \\
\hline
\bullet(X', \blacksquare A \triangleright Y) \triangleright Z
\end{array}$$

 $\dashv$ 

Lemma 5.6 If 
$$\vdash_{\mathbf{DBiKt}} \pi : \circ X, Y \triangleright Z \text{ then } \vdash_{\mathbf{DBiKt}} \pi' : X \triangleright \bullet (Y \triangleright Z).$$

**Proof.** The proof is by induction on  $|\pi|$ . As with the proof of Lemma 5.5, the only intersting cases are when  $\pi$  ends with a propagation rule that moves a formula across X, Y or Z. We show here one case; the others can be proved similarly. Suppose  $X = (X', \blacksquare A)$  and  $\pi$  is as given below left. Then  $\pi'$  is as given below right, where  $\pi'_1$  is obtained by applying the induction hypothesis to  $\pi_1$ .

$$\frac{A, \circ (X', \blacksquare A), Y \triangleright Z}{\circ (X', \blacksquare A), Y \triangleright Z} \blacksquare_{L1} \qquad \rightsquigarrow \qquad \frac{X', \blacksquare A \triangleright \bullet (A, Y \triangleright Z))}{X', \blacksquare A \triangleright \bullet (A \triangleright (Y \triangleright Z))} \xrightarrow{\text{Lm 5.4.iv}} L_{L2}$$

 $\dashv$ 

The following two lemmas are symmetric to Lemma 5.5 and Lemma 5.6 and can be proved similarly.

Lemma 5.7 If 
$$\vdash_{\mathbf{DBiKt}} \pi : X \triangleright Y, \circ Z \ then \vdash_{\mathbf{DBiKt}} \pi' : \bullet(X \triangleright Y) \triangleright Z.$$

Lemma 5.8 If 
$$\vdash_{\mathbf{DBiKt}} \pi : \bullet X, Y \triangleright Z \text{ then } \vdash_{\mathbf{DBiKt}} \pi' : X \triangleright \circ (Y \triangleright Z).$$

We are now ready to prove the main lemma about admissibility of residuation rules.

Lemma 5.9 (Admissibility of residuation) The following statements hold in DBiKt:

- (i) Admissibility of  $rp_{\bullet}$ .  $If \vdash_{\mathbf{DBiKt}} \pi : X \triangleright \bullet Z \ then \vdash_{\mathbf{DBiKt}} \pi' : \circ X \triangleright Z$ .
- (ii) Admissibility of  $rp_{\bullet}$ .  $If \vdash_{\mathbf{DBiKt}} \pi : \circ X \triangleright Z \ then \vdash_{\mathbf{DBiKt}} \pi' : X \triangleright \bullet Z$ .
- (iii) Admissibility of  $rp_{\circ}$ . If  $\vdash_{\mathbf{DBiKt}} \pi : X \triangleright \circ Z \text{ then } \vdash_{\mathbf{DBiKt}} \pi' : \bullet X \triangleright Z$ .

(iv) Admissibility of  $rp_{\circ}$ . If  $\vdash_{\mathbf{DBiKt}} \pi : \bullet X \triangleright Z \ then \vdash_{\mathbf{DBiKt}} \pi' : X \triangleright \circ Z$ .

**Proof.** This is straightforward given Lemma 5.5 - 5.8. We outline the proofs for item i and ii; the rest can be proved symmetrically.

- (i): The only interesting cases are when  $\pi$  ends with a propagation rule that moves a formula into or out of  $\bullet Z$ .
- \* Suppose  $Z = (Z', \Diamond A)$  and  $\pi$  is as given below left. Lemma 5.5 gives  $\pi_2$  from  $\pi_1$ , and  $\pi'$  is as below right:

$$\frac{X \triangleright A, \bullet(Z, \lozenge A)}{X \triangleright \bullet(Z, \lozenge A)} \lozenge_{R1} \qquad \sim \qquad \frac{\circ (X \triangleright A) \triangleright Z, \lozenge A}{\circ (X) \triangleright Z, \lozenge A} \lozenge_{R2}$$

\* Suppose  $X = (X', \blacksquare A)$  and  $\pi$  is as given below left. Then  $\pi'$  is shown below right, where  $\pi'_1$  is the result of applying the induction hypothesis to  $\pi_1$ .

$$\frac{X', \blacksquare A \triangleright \bullet (A \triangleright Z)}{X', \blacksquare A \triangleright \bullet (Z)} \blacksquare_{L2} \qquad \sim \qquad \frac{\pi'_1}{\underbrace{A, \circ (X', \blacksquare A) \triangleright (A \triangleright Z)}_{\circ (X', \blacksquare A) \triangleright Z}} = \text{Lemma 5.4(ii)}$$

- (ii): The only interesting cases are when  $\pi$  ends with a propagation rule that moves a formula into or out of  $\circ X$ .
- \* Suppose  $X = (X', \blacksquare A)$  and  $\pi$  is as given below left. Lemma 5.6 gives  $\pi_2$  from  $\pi_1$ , and  $\pi'$  is as below right:

$$\frac{\pi_1}{\circ (X', \blacksquare A), A \triangleright Z} \bullet_{L1} \quad \sim \quad \frac{\pi_2}{X', \blacksquare A \triangleright \bullet (A \triangleright Z)} \bullet_{L2}$$

\* Suppose  $Z = (Z', \Diamond A)$  and  $\pi$  is as given below left. Then  $\pi'$  is shown below right, where  $\pi'_1$  is the result of applying the induction hypothesis to  $\pi_1$ .

$$\frac{\sigma_{1}}{\circ(X \triangleright A) \triangleright Z', \Diamond A} \diamond_{R2} \qquad \sim \qquad \frac{\pi'_{1}}{(X \triangleright A) \triangleright \bullet(Z', \Diamond A)} - \text{Lemma 5.4(i)}$$

$$\frac{(X \triangleright A) \triangleright \bullet(Z', \Diamond A)}{X \triangleright A, \bullet(Z', \Diamond A)} \diamond_{R1}$$

 $\dashv$ 

#### Admissibility of general contraction

To prove the admissibility of structure contraction, we need to prove several distribution properties among structural connectives. These are stated in the following four lemmas.

**Lemma 5.10** If  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma^+[\circ(X \triangleright Y), \circ Y]$  and contraction on structures is admissible for all derivations  $\pi_1$  such that  $|\pi_1| \leq |\pi|$  then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma^+[\circ(X \triangleright Y)]$ .

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**Proof.** By induction on the height of  $\pi$ . The interesting cases are when  $\pi$  ends with a propagation rule that moves a formula into either  $\circ(X \triangleright Y)$  or  $\circ Y$ :

\* Suppose  $\pi$  ends as below left. Then by Lemma 5.4(ii), there is a derivation  $\pi_2$  of  $\Box A \triangleright \circ (A, X \triangleright Y), \circ Y$  such that  $|\pi_2| \leq |\pi_1|$ . Then we can apply the induction hypothesis to  $\pi_2$  to obtain a derivation  $\pi_3$  of  $\Box A \triangleright \circ (A, X \triangleright Y)$ . Then the derivation below right gives the required:

$$\frac{ \begin{array}{c} \pi_1 \\ \square A \triangleright \circ (A \triangleright (X \triangleright Y)), \circ Y \\ \square A \triangleright \circ (X \triangleright Y), \circ Y \end{array} \square_{L2} \qquad \frac{ \begin{array}{c} \pi_3 \\ \square A \triangleright \circ (A, X \triangleright Y) \\ \square A \triangleright \circ (A \triangleright (X \triangleright Y)) \end{array}}{ \begin{array}{c} \square A \triangleright \circ (A, X \triangleright Y) \\ \square A \triangleright \circ (X \triangleright Y) \end{array} \square_{L2} \qquad 5.4(iv)$$

\* Suppose  $\pi$  ends as below left. Then applying Lemma 5.2 twice, we obtain a derivation  $\pi_2$  of  $\Box A \triangleright \circ (A, X \triangleright Y), \circ (A, X \triangleright Y)$  such that  $|\pi_2| \leq |\pi_1|$ . Then we apply the assumption of this lemma to  $\pi_2$  to obtain a derivation  $\pi_3$  of  $\Box A \triangleright \circ (A, X \triangleright Y)$ . Then the derivation below right gives the required:

$$\frac{\pi_{1}}{\Box A \triangleright \circ (X \triangleright Y), \circ (A \triangleright Y)} \Box_{L2} \qquad \frac{\pi_{3}}{\Box A \triangleright \circ (A, X \triangleright Y)} = \text{Lemma 5.4(iv)}$$

$$\frac{\Box A \triangleright \circ (X \triangleright Y), \circ Y}{\Box A \triangleright \circ (A \triangleright (X \triangleright Y))} \Box_{L2}$$

 $\dashv$ 

Lemma 5.11 If 
$$\vdash_{\mathbf{DBiKt}} \pi : \Sigma^{-}[\circ(X \triangleright Y), \circ X]$$
 then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma^{-}[\circ(X \triangleright Y)]$ .

**Proof.** By induction on the height of  $\pi$ . The interesting cases are when  $\pi$  ends with a propagation rule that moves a formula into either  $\circ(X \triangleright Y)$  or  $\circ X$ :

\* Suppose  $\pi$  ends as below left. Then by Lemma 5.4(i), there is a derivation  $\pi_2$  of  $\circ(X \triangleright Y, A), \circ X \triangleright \lozenge A$  such that  $|\pi_2| \leq |\pi_1|$ . Then we can apply the induction hypothesis to  $\pi_2$  to obtain a derivation  $\pi_3$  of  $\circ(X \triangleright Y, A) \triangleright \lozenge A$ . Then the derivation below right gives the required:

$$\frac{(X \triangleright Y) \triangleright A), \circ X \triangleright \Diamond A}{\circ (X \triangleright Y), \circ X \triangleright \Diamond A} \diamond_{R2} \qquad \frac{(X \triangleright Y, A) \triangleright \Diamond A}{\circ ((X \triangleright Y) \triangleright A) \triangleright \Diamond A} \leftarrow \text{Lemma 5.4(iii)}$$

\* Suppose  $\pi$  ends as below left. Then applying Lemma 5.2 twice, we obtain a derivation  $\pi_2$  of  $\circ(X \triangleright Y, A), \circ(X \triangleright Y, A) \triangleright \Diamond A$  such that  $|\pi_2| \leq |\pi_1|$ . Then we apply the assumption of this lemma to  $\pi_2$  to obtain a derivation  $\pi_3$  of  $\circ(X \triangleright Y, A) \triangleright \Diamond A$ . Then the derivation below right gives the required:

$$\frac{\sigma_1}{\circ (X \triangleright Y), \circ (X \triangleright A) \triangleright \Diamond A} \diamond_{R2} \qquad \frac{\sigma_3}{\circ (X \triangleright Y), \circ X \triangleright \Diamond A} \diamond_{R2} \qquad \frac{\sigma(X \triangleright Y, A) \triangleright \Diamond A}{\circ ((X \triangleright Y) \triangleright A) \triangleright \Diamond A} \diamond_{R2} \qquad \frac{\sigma(X \triangleright Y, A) \triangleright \Diamond A}{\circ (X \triangleright Y) \triangleright \Diamond A} \diamond_{R2}$$

 $\dashv$ 

Lemma 5.12 If 
$$\vdash_{\mathbf{DBiKt}} \pi : \Sigma^+[\bullet(X \triangleright Y), \bullet(Y)]$$
 then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma^+[\bullet(X \triangleright Y)]$ .

Lemma 5.13 If 
$$\vdash_{\mathbf{DBiKt}} \pi : \Sigma^{-}[\bullet(X \triangleright Y), \bullet(X)]$$
 then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma^{-}[\bullet(X \triangleright Y)]$ .

Lemma 5.14 (Admissibility of general contraction) For any structure Y: if  $\vdash_{\mathbf{DBiKt}} \pi : \Sigma[Y,Y]$  then  $\vdash_{\mathbf{DBiKt}} \pi' : \Sigma[Y]$ .

**Proof.** By induction on the size of Y, with a sub-induction on  $|\pi|$ .

- \* For the base case, use Lemma 5.3.
- \* For the case where Y is a  $\triangleright$ -structure, we show the sub-case where Y in a negative context, the other case is symmetric:

$$\begin{array}{l} \frac{\Sigma[(Y_1 \rhd Y_2), (Y_1 \rhd Y_2) \rhd Z]}{\Sigma[Y_1, (Y_1 \rhd Y_2) \rhd Y_2, Z]} - \text{Lemma 5.4(i)} \\ \frac{\Sigma[Y_1, Y_1 \rhd Y_2, Y_2, Z]}{\Sigma[Y_1, Y_1 \rhd Y_2, Y_2, Z]} - \text{Lemma 5.4(i)} \\ \frac{\Sigma[Y_1 \rhd Y_2, Y_2, Z]}{\Sigma[Y_1 \rhd Y_2, Z]} - \text{IH} \\ \frac{\Sigma[Y_1 \rhd Y_2, Z]}{\Sigma[(Y_1 \rhd Y_2) \rhd Z]} - \text{Lemma 5.4(iii)} \end{array}$$

- \* For the case where Y is a  $\circ-$  or  $\bullet$ -structure and  $\pi$  ends with a propagation rule applied to Y, there are three non-trivial sub-cases:
  - · A formula is propagated into Y and Y is in a positive context, as below left. Then by Lemma 5.10, there is a derivation  $\pi'_1$  of  $\Box A, X \triangleright \circ (A \triangleright Z)$ . Then the derivation below right gives the required:

$$\frac{\pi_1}{\Box A, X \rhd \circ (A \rhd Z), \circ Z} \Box_{L2} \qquad \frac{\pi'_1}{\Box A, X \rhd \circ (A \rhd Z)} \Box_{L2}$$

· A formula is propagated into Y and Y is in a negative context, as below left. Then by Lemma 5.11, there is a derivation  $\pi'_1$  of  $\circ(Z \triangleright A) \triangleright X, \Diamond A$ . Then the derivation below right gives the required:

$$\frac{\pi_1}{\circ (Z \triangleright A), \circ Z \triangleright X, \Diamond A} \diamond_{R2} \qquad \frac{\pi'_1}{\circ Z, \circ Z \triangleright X, \Diamond A} \diamond_{R2}$$

· A formula is propagated out of Y, as below left. In this case we use the sub-induction hypothesis to obtain a derivation  $\pi'_1$  of  $X \triangleright A$ ,  $\circ(\blacklozenge A, Z)$ . Then the derivation below right gives the required:

$$\frac{\pi_1}{X \triangleright A, \circ(\blacklozenge A, Z), \circ(\blacklozenge A, Z)} \blacklozenge_{R1} \qquad \frac{\pi'_1}{X \triangleright A, \circ(\blacklozenge A, Z), \circ(\blacklozenge A, Z)} \blacklozenge_{R1}$$

 $\dashv$ 

Once all structural rules of **LBiKt** are shown admissible in **DBiKt**, it is easy to show that every derivation in **LBiKt** can be translated to a derivation in **DBiKt**.

**Theorem 5.15** For any X and Y, if  $\vdash_{\mathbf{LBiKt}} \pi : X \triangleright Y$  then  $\vdash_{\mathbf{DBiKt}} \pi' : X \triangleright Y$ .

**Proof.** By induction on  $|\pi|$ , where  $\pi'_1$  ( $\pi'_2$ ) is obtained from  $\pi_1$  ( $\pi_2$ ) using the IH. We show some cases where  $\pi$  ends in logical rule applications and some where  $\pi$  ends in structural rule applications. The other interesting cases are similar and use Lemmas 5.4 and Lemmas 5.9.

$$\frac{X \triangleright A, Y \qquad X, B \triangleright Y}{X, A \rightarrow B \triangleright Y} \rightarrow_{L} \qquad \sim \\
\frac{X \triangleright A, Y \qquad X, B \triangleright Y}{\pi_{1}''} \rightarrow_{L} \qquad \sim \\
\frac{X \triangleright A, Y \qquad X, A \rightarrow B \triangleright A, Y \qquad Lm. 5.2}{(X, A \rightarrow B \triangleright A) \triangleright Y} \rightarrow_{L} \qquad \sim \\
\frac{X, B \triangleright Y \qquad X, B \triangleright Y}{X, A \rightarrow B, B \triangleright Y} \rightarrow_{L} \qquad \cdots \\
\frac{X, Y \triangleright Z \qquad X, A \rightarrow B \triangleright Y}{X, A \rightarrow B, B \triangleright Y} \rightarrow_{L} \qquad \cdots \\
\frac{X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad Lemma 5.4(iv)}{X, A \rightarrow B \triangleright Y} \rightarrow_{L} \qquad \cdots \\
\frac{\pi_{1} \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad Lemma 5.4(iv)}{X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad Lemma 5.4(iv)} \rightarrow_{L} \qquad \cdots \\
\frac{\pi_{1} \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad Lemma 5.4(iv)}{X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad Lemma 5.9.iv} \rightarrow_{L} \qquad \cdots \\
\frac{\pi_{1} \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad X, Y \triangleright Z \qquad Lemma 5.9.iv}{X, Y \triangleright Z \qquad X, Y$$

**Theorem 5.16** For any X and Y,  $\vdash_{\mathbf{LBiKt}} \pi : X \triangleright Y$  if and only if  $\vdash_{\mathbf{DBiKt}} \pi' : X \triangleright Y$ .

**Proof.** By Theorems 5.1 and 5.15.  $\dashv$ 

# 6 Proof Search

In this section we outline a proof search strategy for **DBiKt**, closely following the approaches presented in [19] and [10]. Here we emphasize the aspects that are new/different because of the interaction between the tense structures  $\circ$  and  $\bullet$  and the intuitionistic structure  $\triangleright$ .

Our proof search strategy proceeds in three stages: saturation, propagation and realisation. The saturation phase applies the "static rules" (i.e. those that do not create extra structural connectives) until further application do not lead to any progress. The propagation phase propagates formulaes across different structural

connectives, while the realisation phase applies the "dynamic rules" (i.e., those that create new structural connectives, e.g.,  $\rightarrow_R$ ).

A context  $\Sigma[\ ]$  is said to be headed by a structural connective # if the topmost symbol in the formation tree of  $\Sigma[\ ]$  is #. A context  $\Sigma[\ ]$  is said to be a factor of  $\Sigma'[\ ]$  if  $\Sigma[\ ]$  is a subcontext of  $\Sigma'[\ ]$  and  $\Sigma[\ ]$  is headed by either  $\triangleright$ ,  $\circ$  or  $\bullet$ . We write  $\widehat{\Sigma}[\ ]$  to denote the least factor of  $\Sigma[\ ]$ . We write  $\widehat{\Sigma}[X]$  to denote the structure  $\Sigma_1[X]$ , if  $\Sigma_1[\ ] = \widehat{\Sigma}[\ ]$ . We define the top-level formulae of a structure as:

$$\{|X|\} = \{A \mid X = (A, Y) \text{ for some } A \text{ and } Y\}.$$

For example, if  $\Sigma[] = (A, B \triangleright C, \bullet(D, (E \triangleright F) \triangleright []))$ , then  $\widehat{\Sigma}[G] = \bullet(D, (E \triangleright F) \triangleright G)$ , and  $\{|D, (E \triangleright F)|\} = \{D\}$ .

Let  $\prec L_1$  and  $\rightarrow_{R_1}$  denote two new derived rules (see [19] for their derivation):

$$\frac{\Sigma^{-}[A, A \prec B]}{\Sigma^{-}[A \prec B]} \prec_{L1} \qquad \frac{\Sigma^{+}[A \rightarrow B, B]}{\Sigma^{+}[A \rightarrow B]} \rightarrow_{R1}$$

We now define a notion of a *saturated structure*, which is similar to that of a traditional sequent. Note that we need to define it for both structures headed by  $\triangleright$  and those headed by  $\circ$  or  $\bullet$ . A structure  $X \triangleright Y$  is *saturated* if it satisfies the following:

- (1)  $\{|X|\} \cap \{|Y|\} = \emptyset$
- (2) If  $A \wedge B \in \{ |X| \}$  then  $A \in \{ |X| \}$  and  $B \in \{ |X| \}$
- (3) If  $A \land B \in \{|Y|\}$  then  $A \in \{|Y|\}$  or  $B \in \{|Y|\}$
- (4) If  $A \vee B \in \{|X|\}$  then  $A \in \{|X|\}$  or  $B \in \{|X|\}$
- (5) If  $A \vee B \in \{|Y|\}$  then  $A \in \{|Y|\}$  and  $B \in \{|Y|\}$
- **(6)** If  $A \to B \in \{|X|\}$  then  $A \in \{|Y|\}$  or  $B \in \{|X|\}$
- (7) If  $A \prec B \in \{ |Y| \}$  then  $A \in \{ |Y| \}$  or  $B \in \{ |X| \}$
- (8) If  $A \prec B \in \{|X|\}$  then  $A \in \{|X|\}$
- (9) If  $A \to B \in \{|Y|\}$  then  $B \in \{|Y|\}$

For structures of the form  $\circ X$  or  $\bullet X$ , we need to define two notions of saturation, left saturation and right saturation. The former is used when  $\circ X$  is nested in a negative context, and the latter when it is in a positive context. A structure  $\circ X$  or  $\bullet X$  is left-saturated if it satisfies (2), (4), (8) above, and

**6'** If  $A \to B \in \{|X|\}$  then  $B \in \{|X|\}$ .

Dually,  $\circ Y$  or  $\bullet Y$  is right-saturated if it satisfies (3), (5), (9) above, and

7' If 
$$A \prec B \in \{|Y|\}$$
 then  $A \in \{|Y|\}$ .

We define structure membership for any two structures X and Y as follows:  $X \in Y$  iff Y = X, X' for some X', modulo associativity and commutativity of comma. For example,  $(A \triangleright B) \in (A, (A \triangleright B), \circ C)$ . The realisation of formulae by a structure X is defined as follows:

\*  $A \to B$  ( $A \subset B$ , resp.) is right-realised (resp. left-realised) by X iff there exists  $Z \triangleright W \in X$  such that  $A \in \{|Z|\}$  and  $B \in \{|W|\}$ .

- \*  $\Box A \ (\Diamond A \text{ resp.})$  is right-realised (resp. left-realised) by X iff there exists  $\circ (Z \triangleright W) \in X$  or  $\circ W \in X$  (resp.  $\circ (W \triangleright Z) \in X$  or  $\circ W \in X$ ) such that  $A \in \{|W|\}$ .
- \*  $\blacksquare A \ ( \blacklozenge A \text{ resp.} )$  is right-realised (resp. left-realised) by X iff there exists  $\bullet (W \triangleright Z) \in X$  or  $\bullet Z \in X$  (resp.  $\bullet (Z \triangleright W) \in X$  or  $\bullet Z \in X$ ) such that  $A \in \{|Z|\}$ .

We say that a structure X is left-realised iff every formula in  $\{|X|\}$  with top-level connective  $-\!\!\!<$ ,  $\Diamond$  or  $\blacklozenge$  is left-realised by X. Right-realisation of X is defined dually. We say that a structure occurrence X in  $\Sigma[X]$  is *propagated* iff no propagation rules are (backwards) applicable to any formula occurrences in X. We define the super-set relation on structures as follows:

- \*  $X_1 \triangleright Y_1 \supset X_0 \triangleright Y_0 \text{ iff } \{ |X_1| \} \supset \{ |X_0| \} \text{ or } \{ |Y_1| \} \supset \{ |Y_0| \}.$
- $* \circ X \supset \circ Y \text{ iff } \bullet X \supset \bullet Y \text{ iff } \{|X|\} \supset \{|Y|\}.$

To simplify presentation, we use the following terminology: Given a structure  $\Sigma[A]$ , we say that  $\widehat{\Sigma}[A]$  is saturated if  $\widehat{\Sigma}[A]$  is  $X \triangleright Y$  and it is saturated; or  $\widehat{\Sigma}[A]$  is either  $\circ X$  or  $\bullet X$  and it is either left- or right-saturated (depending on its position in  $\Sigma[A]$ ). We say that  $\widehat{\Sigma}[A]$  is propagated if its occurrence in  $\Sigma[A]$  is propagated, and we say that A is realised by  $\widehat{\Sigma}[A]$ , if either

- \*  $\widehat{\Sigma}[A] = (X \triangleright Y)$  and either  $A \in \{X\}$  is left-realised by X, or  $A \in \{Y\}$  is right realised by Y; or
- \*  $\widehat{\Sigma}[A]$  is either  $\circ X$  or  $\bullet X$ , and, depending on the polarity of  $\Sigma[\ ]$ , A is either left-or right-realised by X.

We now outline an approach to proof search in  $\mathbf{DBiKt}$ . We approach this by modifying  $\mathbf{DBiKt}$  to obtain a calculus  $\mathbf{DBiKt_1}$  that is more amenable to proof search. Our approach follows that of our previous work on bi-intuitionistic logic [19] since we define syntactic restrictions on rules to enforce a search strategy. For example, we stipulate that a structure must be saturated and propagated before child structures can be created using the  $\rightarrow_R$  rule (see condition ii of Definition 6.1). Additionally and more importantly, our proof search calculus addresses the issue that some modal propagation rules of  $\mathbf{DBiKt}$ , e.g.  $\square_{L2}$ , create  $\triangleright$ -structures during backward proof search. This property of  $\mathbf{DBiKt}$  is undesirable and gives rise to non-termination if rules  $\square_{L2}$  like are applied naively.

**Definition 6.1** Let  $DBiKt_1$  be the system obtained from DBiKt with the following changes:

- (i) Add the derived rules  $\prec L_1$  and  $\rightarrow_{R_1}$ .
- (iii) Replace rules  $\triangleright_{L1}$  and  $\triangleright_{R1}$  with the following:

$$\frac{\Sigma[A, (A, X \triangleright Y), W \triangleright Z]}{\Sigma[(A, X \triangleright Y), W \triangleright Z]} \triangleright_{L1} \qquad \frac{\Sigma[W \triangleright Z, (X \triangleright Y, A), A]}{\Sigma[W \triangleright Z, (X \triangleright Y, A)]} \triangleright_{R1}$$

(iv) Restrict rules  $\triangleright_{L2}$  and  $\triangleright_{R2}$  with the following condition: the rule is applicable only if  $A \notin \{|Y|\}$ .

(v) Replace rules  $\Diamond_L$ ,  $\Box_R$ ,  $\blacklozenge_L$ ,  $\blacksquare_R$  with the following, where the rule is applicable only if  $\widehat{\Sigma}[\#A]$  is saturated and propagated and #A is not realised by  $\widehat{\Sigma}[\#A]$ , for  $\# \in \Diamond, \Box, \blacklozenge, \blacksquare$ :

$$\frac{\Sigma^{-}[\lozenge A, \circ (A \triangleright \emptyset)]}{\Sigma^{-}[\lozenge A]} \lozenge_{L} \qquad \frac{\Sigma^{+}[\Box A, \circ (\emptyset \triangleright A)]}{\Sigma^{+}[\Box A]} \Box_{R}$$

$$\Sigma^{-}[\lozenge A, \circ (A \triangleright \emptyset)] \qquad \qquad \Sigma^{+}[\Box A, \circ (\emptyset \triangleright A)]$$

$$\frac{\Sigma^{-}[\phi A, \bullet(A \triangleright \emptyset)]}{\Sigma^{-}[\phi A]} \phi_{L} \qquad \frac{\Sigma^{+}[\blacksquare A, \bullet(\emptyset \triangleright A)]}{\Sigma^{+}[\blacksquare A]} \blacksquare_{R}$$

(vi) Replace rules  $\blacksquare_{L2}, \Box_{L2}$  with the following, where  $A \notin \{Y_1\}$ :

$$\frac{\Sigma[\blacksquare A, X \rhd \bullet (A, Y_1 \rhd Y_2), Z]}{\Sigma[\blacksquare A, X \rhd \bullet (Y_1 \rhd Y_2), Z]} \blacksquare_{L2} \qquad \frac{\Sigma[\Box A, X \rhd \circ (A, Y_1 \rhd Y_2), Z]}{\Sigma[\Box A, X \rhd \circ (Y_1 \rhd Y_2), Z]} \Box_{L2}$$

(vii) Replace rules  $\Diamond_{R2}, \blacklozenge_{R2}$  with the following, where  $A \notin \{|X_2|\}$ :

$$\frac{\Sigma[\circ(X_1 \triangleright X_2, A), Y \triangleright Z, \lozenge A]}{\Sigma[\circ(X_1 \triangleright X_2), Y \triangleright Z, \lozenge A]} \diamondsuit_{R2} \qquad \frac{\Sigma[\bullet(X_1 \triangleright X_2, A), Y \triangleright Z, \blacklozenge A]}{\Sigma[\bullet(X_1 \triangleright X_2), Y \triangleright Z, \blacklozenge A]} \blacklozenge_{R2}$$

(viii) Replace rules  $\Box_{L1}, \Diamond_{R1}, \blacksquare_{L1}, \blacklozenge_{R1}$  with the following:

$$\frac{\Sigma^{-}[A, \bullet(\Box A, X \triangleright Y)]}{\Sigma^{-}[\bullet(\Box A, X \triangleright Y)]} \,\Box_{L1} \qquad \frac{\Sigma^{+}[A, \bullet(Y \triangleright \lozenge A, X)]}{\Sigma^{+}[\bullet(Y \triangleright \lozenge A, X)]} \,\lozenge_{R1}$$

$$\frac{\Sigma^{-}[A, \circ(\blacksquare A, X \triangleright Y)]}{\Sigma^{-}[\circ(\blacksquare A, X \triangleright Y)]} \blacksquare_{L1} \qquad \frac{\Sigma^{+}[A, \circ(Y \triangleright \spadesuit A, X)]}{\Sigma^{+}[\circ(Y \triangleright \spadesuit A, X)]} \spadesuit_{R1}$$

(ix) Replace rules  $\rightarrow_L$ ,  $\prec R$  with the following:

$$\frac{\Sigma[X,A\to B\rhd A,Y] \qquad \Sigma[X,A\to B,B\rhd Y]}{\Sigma[X,A\to B\rhd Y]}\to_L$$

$$\frac{\Sigma[X \triangleright Y, A \multimap B, A] \qquad \Sigma[X, B \triangleright Y, A \multimap B]}{\Sigma[X \triangleright Y, A \multimap B]} \multimap_R$$

(x) Restrict rules  $\to_L$ ,  $\prec$  R,  $\triangleright_{L1}$ ,  $\triangleright_{R1}$ ,  $\land_L$ ,  $\land_R$ ,  $\lor_L$ ,  $\lor_R$  and all modal propagation rules to the following: Let  $\Sigma[X_0]$  be the conclusion of the rule and let  $\Sigma[X_1]$  (and  $\Sigma[X_2]$ ) be the premise(s). The rule is applicable only if:  $\widehat{\Sigma}[X_1] \supset \widehat{\Sigma}[X_0]$  and  $\widehat{\Sigma}[X_2] \supset \widehat{\Sigma}[X_0]$ .

We conjecture that  $\mathbf{DBiKt}$  and  $\mathbf{DBiKt_1}$  are equivalent and that backward proof search in  $\mathbf{DBiKt}$  terminates. Note that by equivalence here we mean that  $\mathbf{DBiKt}$  and  $\mathbf{DBiKt}$  proves the same set of formulae, but not necessarily the same set of structures. This is because the propagation rules in  $\mathbf{DBiKt_1}$  are more restricted so as to allow for easier termination checking. For example, the structure  $A \triangleright \bullet (\lozenge A)$  is derivable in  $\mathbf{DBiKt}$  but not in  $\mathbf{DBiKt_1}$ , although its formula translate is derivable in both systems. It is likely that a combination of the techniques from [19] and [10] can be used to prove termination of proof search in  $\mathbf{DBiKt_1}$ , given its similarities to the deep inference systems used in those two works.

### Goré, Postniece and Tiu

```
w \Vdash \top
                                  for every w \in W
                                                                                               w \Vdash \bot
                                                                                                                               for no w \in W
w \Vdash A \land B if w \Vdash A \& w \Vdash B
                                                                                               w \Vdash A \lor B if w \Vdash A or w \Vdash B
w \Vdash A \to B if \forall v > w \ v \Vdash A \Rightarrow v \Vdash B
                                                                                               w \Vdash \neg A
                                                                                                                       if \forall v > w. \ v \not\Vdash A
w \Vdash A \mathrel{-\!\!\!\!\!-\!\!\!\!\!-} B \quad \text{if} \quad \exists v \leq w. \ v \Vdash A \ \& \ v \not\Vdash B
                                                                                               w \Vdash \sim A
                                                                                                                       if \exists v \leq w. \ v \not\Vdash A
w\Vdash \Diamond A
                           if \exists v. \ wR_{\Diamond}v \& v \Vdash A
                                                                                               w \Vdash \Box A
                                                                                                                       if \forall z. \forall v. \ w \leq z \ \& \ zR_{\square}v \Rightarrow v \Vdash A
                            if \exists v. \ wR_{\square}^{-1}v \ \& \ v \Vdash A
                                                                                               w\Vdash \blacksquare A
                                                                                                                       if \forall z. \forall v. \ w \leq z \ \& \ zR_{\wedge}^{-1}v \Rightarrow v \Vdash A
w\Vdash \blacklozenge A
```

Fig. 5. Semantics for BiKt

### 7 Semantics

We now give a Kripke-style semantics for BiKt and show that  $\mathbf{LBiKt}$  is sound with respect to the semantics. Our semantics for BiKt extend Rauszer's [21] Kripke-style semantics for BiInt by clauses for the tense logic connectives. We use the classical first-order meta-level connectives &, "or", "not",  $\Rightarrow$ ,  $\forall$  and  $\exists$  to state our semantics.

A Kripke frame is a tuple  $\langle W, \leq, R_{\Diamond}, R_{\Box} \rangle$  where W is a non-empty set of worlds and  $\leq \subseteq (W \times W)$  is a reflexive and transitive binary relation over W, and each of  $R_{\Diamond}$  and  $R_{\Box}$  are arbitrary binary relations over W with the following frame conditions:

```
F1\Diamond if x \leq y \& xR_{\Diamond}z then \exists w. yR_{\Diamond}w \& z \leq w
F2\Box if xR_{\Box}y \& y \leq z then \exists w. x \leq w \& wR_{\Box}z.
```

A Kripke model extends a Kripke frame with a mapping V from Atoms to  $2^{W}$  obeying persistence:

$$\forall v > w. \ w \in V(p) \Rightarrow v \in V(p).$$

Given a model  $\langle W, \leq, R_{\Diamond}, R_{\square}, V \rangle$ , we say that  $w \in W$  satisfies p if  $w \in V(p)$ , and write this as  $w \Vdash p$ . We write  $w \not\Vdash p$  to mean  $(not)(w \Vdash p)$ ; that is,  $\exists v \geq w. \ v \not\in V(p)$ . The relation  $\Vdash$  is then extended to formulae as given in Figure 5. A BiKt-formula A is BiKt-valid if it is satisfied by every world in every Kripke model. A nested sequent  $X \triangleright Y$  is BiKt-valid if its formula translation is BiKt-valid.

Our semantics differ from those of Simpson [23] and Ewald [6] because we use two modal accessibility relations instead of one. In our calculi, there is no direct relationship between  $\lozenge$  and  $\square$  (or  $\blacklozenge$  and  $\blacksquare$ ), but  $\lozenge$  and  $\blacksquare$  are a residuated pair, as are  $\blacklozenge$  and  $\square$ . Semantically, this corresponds to  $R_{\blacklozenge} = R_{\square}^{-1}$  and  $R_{\blacksquare} = R_{\lozenge}^{-1}$ ; therefore the clauses in Figure 5 are couched in terms of  $R_{\lozenge}$  and  $R_{\square}$  only. Our frame conditions F1 $\lozenge$  and F2 $\square$  are also used by Simpson whose F2 captures the "persistence of being seen by" [23, page 51] while for us F2 $\square$  is simply the "persistence of  $\spadesuit$ ".

LBiKt is sound with respect to BiKt. The soundness proof is straightforward by the definition of the semantics and the inference rules.

**Theorem 7.1 (Soundness)** If A is a BiKt-formula and  $\emptyset \triangleright A$  is LBiKt-derivable, then A is BiKt-valid.

We conjecture that **DBiKt** is also complete w.r.t. the semantics. We give an outline here: The proof follows the usual counter-model construction technique for intuitionistic and tense logics; the non-trivial addition is showing that the resulting models satisfy the frame conditions  $F1\lozenge$  and  $F2\square$ . We will show that our propagation rules allow us to simulate the frame conditions. We do the case for  $F1\diamondsuit$ ; the other is similar.

We need to consider all the derivation fragments ending with a deep sequent

such that it contains the syntactic equivalent of three worlds x, y and z such that  $x \leq y$  and  $xR_{\Diamond}z$ . Then we need to show that there exists a w such that  $yR_{\Diamond}w$  and  $z \leq w$ . We will take z to be the required w, which means we need to show that  $yR_{\Diamond}z$  and  $z \leq z$ . The latter follows immediately because  $\leq$  is reflexive. The former demands the following:

- (i) If  $y \not\Vdash \Diamond B$  then  $z \not\Vdash B$
- (ii) If  $z \Vdash \blacksquare A$  then  $y \Vdash A$

The following derivation fragments illustrate the required propagations for (1) on the left and (2) on the right:

$$\frac{\Sigma[(\circ(Z \triangleright B) \triangleright U, \Diamond B) \triangleright Y, \Diamond B]}{\Sigma[(\circ Z \triangleright U, \Diamond B) \triangleright Y, \Diamond B]} \Diamond_{R2} \\ \frac{\Sigma[(\circ Z \triangleright U, \Diamond B) \triangleright Y, \Diamond B]}{\Sigma[(\circ Z \triangleright U) \triangleright Y, \Diamond B]} \triangleright_{R2} \\ \frac{\Sigma[(\circ(BA, Z) \triangleright X) \triangleright U]}{\Sigma[(\circ(BA, Z) \triangleright X) \triangleright U]} \bullet_{L1} \\ \frac{\Sigma[\circ(Z \triangleright B) \triangleright (Y_1 \triangleright Y_2, \Diamond B), \Diamond B]}{\Sigma[\circ Z \triangleright (Y_1 \triangleright Y_2, \Diamond B), \Diamond B]} \Diamond_{R2} \\ \frac{\Sigma[\circ Z \triangleright (Y_1 \triangleright Y_2, \Diamond B), \Diamond B]}{\Sigma[\circ Z \triangleright (Y_1 \triangleright Y_2, \Diamond B)]} \triangleright_{R1} \\ \frac{\Sigma[Z \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]}{\Sigma[Z \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]} \Diamond_{R1} \\ \frac{\Sigma[Z \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]}{\Sigma[Z \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]} \triangleright_{R1} \\ \frac{\Sigma[A \triangleright \bullet(A \triangleright (A, Y_1 \triangleright Y_2))]}{\Sigma[BA \triangleright \bullet(A \triangleright (Y_1 \triangleright Y_2))]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2, \Diamond B), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet(Y_1 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]}{\Sigma[BA \triangleright \bullet((Y_1 \triangleright Y_2), \Diamond B)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_1), (X_2 \triangleright Y_2)]}{\Sigma[BA \triangleright \bullet((X_1 \triangleright Y_1), (X_2 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_1), (X_2 \triangleright Y_2)]}{\Sigma[BA \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]}{\Sigma[BA \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]}{\Sigma[BA \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]}{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]}{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2 \triangleright Y_2)]} \bullet_{L2} \\ \frac{\Sigma[A \triangleright \bullet((X_1 \triangleright Y_2), (X_2$$

# 8 Modularity, Extensions and Classicality

We first exhibit the modularity of our deep calculus  $\mathbf{DBiKt}$  by showing that fragments of  $\mathbf{DBiKt}$  obtained by restricting the language of formulae and structures also satisfy cut admissibility. For example, by allowing only Int formulae and structures, and restricting to rules that only affect those structures, we get a subsystem of  $\mathbf{DBiKt}$  that admits cut admissibility. We then show how we can obtain Ewald's intuitionistic tense logic IKt [6], Simpson's intuitionistic modal logic IK [23] and regain classical tense logic Kt. We also discuss extensions of  $\mathbf{DBiKt}$  with axioms T, 4 and B but they do not correspond semantically to reflexivity, transitivity and symmetry [23].

### **Modularity**

A nested sequent is *purely modal* if contains no occurrences of  $\bullet$  nor its formula translates  $\blacksquare$  and  $\blacklozenge$ . We write **DInt** for the sub-system of **DBiKt** containing only the rules id, the logical rules for intuitionistic connectives, and the propagation rules for  $\triangleright$ . The logical system **DIntK** is obtained by adding to **DInt** the deep introduction rules for  $\square$  and  $\diamondsuit$ , and the propagation rules  $\square_{L2}$  and  $\diamondsuit_{R2}$ . The logical system **DBInt** is obtained by adding to **DInt** the deep introduction rules for  $\multimap$ . In the following, we say that a formula is an IntK-formula if it is composed from propositional variables, intuitionistic connectives, and  $\square$  and  $\diamondsuit$ . Observe that in

Fig. 6. Derivations of Simpson's axiom 2 and Ewald's axiom 5 (left) and Ewald's axiom 7 (right)

Fig. 7. Derivations of Simpson's axiom 5 and Ewald's axiom 10 (left) and Ewald's axiom 11' (right)

**DBiKt**, the only rules that create • upwards are  $\blacklozenge_L$  and  $\blacksquare_R$ . Thus in every **DBiKt**-derivation  $\pi$  of an IntK formula, the internal sequents in  $\pi$  are purely modal, and hence  $\pi$  is also a **DIntK**-derivation. This observation gives immediately the following modularity result.

**Theorem 8.1 (Modularity)** Let A be an Int (resp. BiInt and IntK) formula. The nested sequent  $\emptyset \triangleright A$  is **DInt**-derivable (resp. **DBInt**- and **DIntK**-derivable) iff  $\emptyset \triangleright A$  is **DBiKt**-derivable.

A consequence of Theorem 4.11, Theorem 5.1, Theorem 5.15 and Theorem 8.1, is that the cut rule is admissible in **DInt**, **DBInt** and **DIntK**. As the semantics of **LBiKt** (hence, also **DBiKt**) is conservative w.r.t. to the semantics of both intuitionistic and bi-intuitionistic logic, the following completeness result holds.

**Theorem 8.2** An Int (resp. BiInt) formula A is valid in Int (resp. BiInt) iff  $\emptyset \triangleright A$  is derivable in **DInt** (resp. **DBInt**).

#### Obtaining Ewald's IKt

To obtain Ewald's IKt [6] we need to collapse  $R_{\Diamond}$  and  $R_{\Box}$  into one temporal relation R and leave out our semantic clauses for  $\prec$  and  $\sim$ . That is, we need to

add the following conditions to the basic semantics:  $R_{\Diamond} \subseteq R_{\Box}$  and  $R_{\Box} \subseteq R_{\Diamond}$ . Proof theoretically, this is captured by extending **LBiKt** with the structural rules:

$$\frac{X \rhd \bullet Y \rhd \bullet Z}{X \rhd \bullet (Y \rhd Z)} \bullet \rhd_R \qquad \frac{X \rhd \circ Y \rhd \circ Z}{X \rhd \circ (Y \rhd Z)} \circ \rhd_R$$

We refer to the extension of LBiKt with these two structural rules as LBiKtE.

Simpson's intuitionistic modal logic IK [23] can then be obtained from Ewald's system by restricting the language to the modal fragment. Note that cut-elimination still holds for **LBiKtE** because these structural rules are closed under formula substitution and the cut-elimination proof for **LBiKt** still goes through when additional structural rules of this kind are added. We refer the reader to [10] for a discussion on how cut elimination can be proved for this kind of extensions.

A BiKt-frame is an E-frame if  $R_{\square} = R_{\lozenge}$ . A formula A is E-valid if it is true in all worlds of every E-model. An IKt formula A is a theorem of IKt iff it is E-valid [6]. The rules  $\circ \triangleright_R$  and  $\bullet \triangleright_R$  are sound for E-frames.

**Lemma 8.3** Rule  $\circ \triangleright_R$  is sound iff  $R_{\square} \subseteq R_{\lozenge}$ .

**Proof.** ( $\Leftarrow$ ) We show that if the frame condition holds, then the rule is sound. We assume that: (1)  $R_{\square} \subseteq R_{\Diamond}$ , and (2) that the formula translation  $\Diamond A \to \square B$  of the premise is valid. We then show that the formula translation  $\square(A \to B)$  of the conclusion is valid. For a contradiction, suppose that  $\square(A \to B)$  is not valid. That is, there exists a world u such that  $u \not \models \square(p \to q)$ . Then (4) there exist worlds x and y such that  $u \leq x \& xR_{\square}y$  and  $y \not \models p \to q$ . Thus there exists z s.t.  $z \geq y$  and  $z \not \models p$  and  $z \not \models q$ . The pattern  $xR_{\square}y \leq z$  implies there is a world w with  $x \leq wR_{\square}z$  by F2 $\square$ . The frame condition (1) then gives  $wR_{\Diamond}z$  too, meaning that  $w \not \models \Diamond p$ . From (2) we get  $w \not \models \square q$ , which gives us  $z \not \models q$ , giving us the contradiction we seek. Therefore the premise  $\square(A \to B)$  is valid and the rule is sound.

(⇒) We show that if the rule is sound, then the failure of the frame condition gives a contradiction. So suppose that the rule is sound. The rule implies that  $\triangleright(\lozenge A \to \square B) \to \square(A \to B)$  is derivable. For a contradiction, suppose we have a frame with  $R_{\square} \not\subseteq R_{\lozenge}$ . That is, (5): there exist x and y such that  $xR_{\square}y$  but not  $xR_{\lozenge}y$ . Let  $W = \{u, w, x, y, z\}$ , let < be the relation  $\{(u, x), (x, w), (y, z)\}$  and let < be the reflexive-transitive closure of <. Let  $R_{\lozenge} = \{\}$ ,  $R_{\square} = \{(x, y), (w, z)\}$  and let  $V(p) = \{z\}, V(q) = \{\}$ . Then the model  $\langle W, \leq, R_{\lozenge}, R_{\square}, V \rangle$  satisfies (5), and has  $u \Vdash \lozenge p \to \square q$  but  $u \not\models \square (p \to q)$ .  $\dashv$ 

**Lemma 8.4** Rule  $\bullet \triangleright_R$  is sound iff  $R_{\Diamond} \subseteq R_{\Box}$ .

**Proof.**  $R_{\Diamond} \subseteq R_{\Box}$  means  $R_{\blacksquare} \subseteq R_{\blacklozenge}$ ; the rest of the proof is analogous to the proof of Lemma 8.3.  $\dashv$ 

**Theorem 8.5** If A is derivable in LBiKtE then A is E-valid.

**Proof.** Straightforward from the soundness of **LBiKt** w.r.t. BiKt-semantics (which subsumes Ewald's semantics) and Lemma 8.3 and Lemma 8.4. ⊢

Completeness of **LBiKtE** w.r.t. IKt and IK can be shown by deriving the axioms of IKt and IK.

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$$\begin{array}{c|c} \Sigma^{-}[A, \Box A] \\ \hline \Sigma^{-}[\Box A] \end{array} T \Box \qquad \begin{array}{c|c} \Sigma[\Box A, X \rhd \circ (\Box A \rhd Y), Z] \\ \hline \Sigma[\Box A, X \rhd \circ Y, Z] \end{array} 4 \Box_L \qquad \begin{array}{c|c} \Sigma^{-}[A, \circ (\Box A, X)] \\ \hline \Sigma^{-}[\circ (\Box A, X)] \end{array} B \Box_L \\ \hline \\ \hline \frac{p, \Box p \rhd p}{\Box p \rhd p} \overset{id}{T} \\ \hline \Box p \rhd \circ (\Box p \rhd \Box p) & 4 \Box_L \\ \hline \Box p \rhd \Box p & \Box R \\ \hline \rhd \Box p \rhd \Box p & \rightarrow_R \end{array} \qquad \begin{array}{c|c} \hline p, \circ \Box p \rhd p & id \\ \hline \Box p \rhd \Box p & \Box R \\ \hline \rhd \Box p \rhd \Box p & \rightarrow_R \end{array} \qquad \begin{array}{c|c} \hline p, \circ \Box p \rhd p & id \\ \hline \odot D p \rhd p & O\Box p & O\Box P \\ \hline \rhd \Box p \rhd \Box p & \rightarrow_R \end{array}$$

Fig. 8. Some example propagation rules and the axioms they capture

Theorem 8.6 System LBiKtE is complete w.r.t. Ewald's IKt and Simpson's IK.

**Proof.** We show the non-trivial cases; the rest are similar or easier. Derivations of Simpson's axiom 2 and Ewald's axiom 5 and 7 are given in Figure 6, derivations of Simpson's axiom 5 and Ewald's axiom 10 and 11' are given in Figure 7.  $\dashv$ 

**Theorem 8.7 (Conservativity over IKt and IK)** If A is an IKt-formula (IK formula), then A is IKt-valid (IK-valid) iff  $\emptyset \triangleright A$  is derivable in LBiKtE.

### Regaining classical tense logic Kt

To collapse BiKt to classical tense logic Kt we add the rules  $\bullet \triangleright_R$  and  $\circ \triangleright_L$ , giving Ewald's IKt with  $R_{\Diamond} = R_{\Box}$  via Lemmas 8.3-8.4, and then add following two rules:

$$\frac{X_1, X_2 \triangleright Y_1, Y_2}{(X_1 \triangleright Y_1), X_2 \triangleright Y_2} s_L^{-1} \qquad \frac{X_1, X_2 \triangleright Y_1, Y_2}{X_1 \triangleright Y_1, (X_2 \triangleright Y_2)} s_R^{-1}$$

The law of the excluded middle and the law of (dual-)contradiction can then be derived as shown below:

$$\frac{\frac{p \triangleright p, \bot}{(\emptyset \triangleright p), p \triangleright \bot} s_L^{-1}}{\frac{(\emptyset \triangleright p) \triangleright (p \rightarrow \bot)}{\emptyset \triangleright p, (p \rightarrow \bot)} s_L} \xrightarrow{S_L} \frac{\frac{p, \top \triangleright p}{\top \triangleright p, (p \triangleright \emptyset)} s_R^{-1}}{\frac{(\top \multimap p) \triangleright (p \triangleright \emptyset)}{\emptyset} s_R} \searrow_L 
\frac{\frac{p, \top \triangleright p}{\top \triangleright p, (p \triangleright \emptyset)} s_R^{-1}}{\frac{(\top \multimap p) \triangleright (p \triangleright \emptyset)}{\emptyset} s_R} \searrow_L 
\frac{p, (\top \multimap p) \triangleright (p \triangleright \emptyset)}{p, (\top \multimap p) \triangleright \emptyset} \land_R$$

### Further extensions

Our previous work on deep inference systems for classical tense logic [10] shows that extensions of classical tense logic with some standard modal axioms can be formalised by adding numerous propagation rules to the deep inference system for classical tense logic given in that paper. We illustrate here with a few examples how such an approach to extensions with modal axioms can be applied to BiKt. Figure 8 shows the propagation rules that are needed to derive axiom T, 4 and B. For each rule, the derivation of the corresponding axiom is given below the rule. Other nesting combinations will be needed for full completeness. Dual rules allow derivations of  $p \to \Diamond p$  and  $\Diamond \Diamond p \to \Diamond p$ . The complete treatement of these and other possible extensions of **LBiKt** is left for future work.

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### References

- [1] G. Amati and F. Pirri. A uniform tableau method for intuitionistic modal logics i. Studia Logica, 53(1):29-60, 1994.
- [2] K Brünnler and L Straßburger Modular Sequent Systems for Modal Logic In Proc. TABLEAUX, LNCS:5607;152-166. Springer, 2009.
- [3] M J Collinson, B. Hilken and D. Rydeheard. Semantics and proof theory of an intuitionistic modal sequent calculus. Technical report, University of Manchester, UK, 1999.
   [4] T. Crolard. A formulae-as-types interpretation of Subtractive Logic. J. of Logic and Comput.,
- 14(4):529-570, 2004.
- [5] R. Davies and F. Pfenning. A modal analysis of staged computation. J. ACM, 48(3):555–604, 2001.
- [6] W. B. Ewald. Intuitionistic tense and modal logic. J. Symb. Log, 51(1):166-179, 1986.
- [7] D. Galmiche and Y. Salhi. Calculi for an intuitionistic hybrid modal logic. In Proc. IMLA, 2008.
- [8] R. Goré. Substructural logics on display. Log. J of Interest Group in Pure and Applied Logic, 6(3):451-504, 1998.
- [9] R. Goré, L. Postniece, and A. Tiu. Cut-elimination and proof-search for bi-intuitionistic logic using nested sequents. In *Proc. AiML* 7:43–66. College Publications, 2008.
- [10] R. Goré, L. Postniece, and A. Tiu. Taming displayed tense logics using nested sequents with deep inference. In Proc. TABLEAUX, LNCS:5607;189–204. Springer, 2009.
- [11] Y. Kakutani. Calculi for intuitionistic normal modal logic. In Proceedings of PPL 2007.
- [12] R. Kashima. Cut-free sequent calculi for some tense logics. Studia Logica, 53:119-135, 1994.
- [13] A. Masini. 2-sequent calculus: Intuitionism and natural deduction. J. Log. Comput., 3(5):533-562, 1993.
- [14] G Mints. On some calculi of modal logic. Proc. Steklov Inst. of Mathematics, 98:97-122, 1971.
- [15] T. Murphy VII, K. Crary, R. Harper, and F. Pfenning. A symmetric modal lambda calculus for distributed computing. In LICS, pages 286–295, 2004.

  [16] F. Pfenning and H.-C. Wong. On a modal lambda calculus for S4. Electr. Notes Theor. Comput. Sci.,
- [17] L. Pinto and T. Uustalu. Proof search and counter-model construction for bi-intuitionistic propositional logic with labelled sequents. In TABLEAUX, pages 295–309, 2009.
- [18] F Poggiolesi. The Tree-hypersequent Method for Modal Propositional Logic. Trends in Logic: Towards Mathematical Philosophy, pp 9–30, Springer, 2009.
- [19] L. Postniece. Deep inference in bi-intuitionistic logic. In *Proc. WoLLIC*, LNCS 5514:320–334. Springer, 2009.
- [20] C. Rauszer. A formalization of the propositional calculus of H-B logic. Studia Logica, 33:23–34, 1974.
- [21] C. Rauszer. An algebraic and Kripke-style approach to a certain extension of intuitionistic logic. Dissertationes Mathematicae, 168, 1980.
- [22] M. Sadrzadeh and R. Dyckhoff. Positive logic with adjoint modalities: Proof theory, semantics and reasoning about information. Electr. Notes in TCS, 249:451–470, 2009.
- [23] A. K. Simpson. The proof theory and semantics of intuitionistic modal logic. PhD thesis, Univ. of Edinburgh, 1994.
- [24] H. Wansing. Sequent calculi for normal modal proposisional logics. J. Logic and Computation, 4(2):125-142, Apr. 1994.